Set Theory

Pawel Garbacz
garbacz@kul.pl
http://pracownik.kul.pl/garbacz/dydaktyka

October 1, 2012

Contents

1 Baby set theory **5**

1.1 Introduction 5

1.2 Further reading 5

1.2.1 Books 5

1.2.2 On-line resources 6

1.3 First steps 6

1.3.1 Why all this pain? 6

1.3.2 Learning set theory alphabet 7

1.3.3 “Hello, world!” in set theory 8

1.3.4 Beyond the basics 12

1.3.5 Families of sets, etc. 15

1.3.6 Relations 22

2 When things go wrong in logic ... **31**

2.1 Set theory dies 31

2.2 ... and it rises from the dead. 33

3 Zermelo-Fraenkel set theory **37**

3.1 Axioms 37

3.2 What about antinomies in ZF theory? 38

3.3 Back to the basics 39

3.4 Cardinal numbers 39

3.4.1 Let’s count! 39

3.4.2 Beyond the finite 40

3.4.3 Infinity comparison 45

3.4.4 How to calculate with cardinal numbers 46

3.4.5 Cantor’s theorem 48

Chapter 1

Baby set theory

1.1 Introduction

1. I have some good news and some bad news.
   (a) The bad news is that set theory is part of mathematics.
   (b) The good news is that set theory one of the easiest part of mathematics.
2. We will require some language from the Formal Logic lecture:
   (a) $\neg, \wedge, \vee, \rightarrow, \equiv$,
   (b) $\forall, \exists$.
3. We will require some understanding of this language:
   (a) truth-table method
   (b) awareness of some laws from predicate logic

1.2 Further reading

1.2.1 Books

- S. Lipschutz, *Schaum's Outline of Set Theory and Related Topics*, McGraw-Hill 1998
- K. Devlin, *The Joy of Sets: Fundamentals of Contemporary Set Theory*, Springer 1993
- T. Jech, *Set theory*, Springer 2006

1.2.2 On-line resources

- `http://plato.stanford.edu/entries/set-theory`
- `http://tedsider.org/teaching/st/st_notes.pdf`
- `www.math.clemson.edu/~mjs/courses/misc/settheory.pdf`
- YouTube's channels
  - `www.youtube.com/user/bullcleo1`

1.3 First steps

1.3.1 Why all this pain?

1. notion of infinity and derivatives
   - Zeno's paradoxes
     (a) Achilles and the tortoise

     > In a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead. (Aristotle, *Physics* VI, 9 239b15)

     (b) dichotomy

     > That which is in locomotion must arrive at the half-way stage before it arrives at the goal. (Aristotle, *Physics* VI, 9 239b10)

     (c) arrow

     > If everything when it occupies an equal space is at rest, and if that which is in locomotion is always occupying such a space at any moment, the flying arrow is therefore motionless.(Aristotle, *Physics* VI, 9 239b5)

2. (medieval problem of) universals
   - nominalism
   - conceptualism
   - realism
3. logical metaphysics (E. Zalta, M. Bunge, N. Goodman)

1.3.2 Learning set theory alphabet

Set theory is about sets.

You can think about sets as if they were:

- collections
- lists
- containers
- bags
- sacks
- bottles
- boxes
- ...

For a set-lover it is of uttermost importance to be able to say that something belongs to a set. So he or she needs to be allowed to say that

- George W. Bush belongs to the set of American presidents.
- The Earth belongs to the set of planets.
- Warsaw belongs to the set of capital cities.

Similarly, he or she wants to negate that something belongs to a set:

- George W. Bush does not belong to the set of capital cities.
- The Earth does not belong to the set of American presidents.
- Warsaw does not belong to the set of planets.

To this end set theory provides him with its most amazing notion: the notion of membership $\in$. Having this symbol we can rewrite the above statements as below:

- George W. Bush $\in$ the set of American presidents.
- The Earth $\in$ the set of planets.
- Warsaw $\in$ the set of capital cities.
- George W. Bush $\notin$ to the set of capital cities.
- The Earth $\notin$ to the set of American presidents.
- Warsaw $\notin$ to the set of planets.

In fact, since all set-theoretical notions are defined only by means of $\in$ (and suitable logical symbols), one can say that set theory is a theory of $\in$.

Set theory is amazing also because it may enhance your artistic imagination. When you want to say that x belongs to set Y, you can draw a circle (or ellipsis, rectangle, etc.) to represent Y and when you draw x inside Y, your drawing will say x is a member of Y. Accordingly, when you place x outside Y, you will say that x is not a member of Y.

Set theory has certain philosophical ramifications. A set can be seen as

1. group of objects
2. concept
3. universal entity

Thus, the set of human beings can be seen as

1. group of people
2. idea/concept of man
3. humanity

1.3.3 "Hello, world!" in set theory

In set theory one can say that one set is included in another

$$X \subseteq Y \triangleq \forall z \ (z \in X \to z \in Y). \tag{1.1}$$

Figure 1.2 shows the graphical interpretation of this relation.

And here some examples:

1. set of American presidents $\subseteq$ set of presidents
2. set of planets $\subseteq$ set of celestial bodies
3. set of monkeys $\subseteq$ set of mammals

In set theory we can also say that one set is a union of two sets:

$$z \in X \cup Y \triangleq z \in X \vee z \in Y. \tag{1.2}$$

And here some examples:

1. set of human beings = set of male human beings $\cup$ set of female human beings
2. set of natural numbers = set of even natural numbers $\cup$ set of odd natural numbers
3. set of cars = set of cars $\cup$ set of Audis

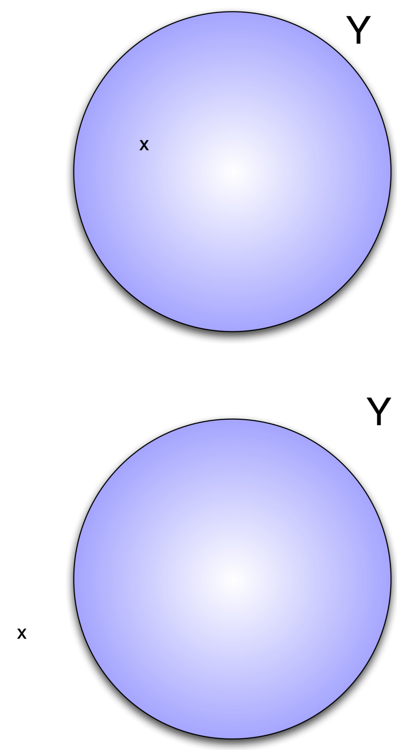

Figure 1.1: Set membership

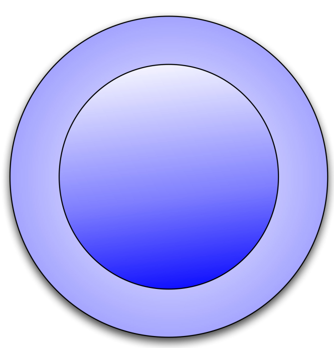

Figure 1.2: Set inclusion

In set theory we can also say that one set is a product of two sets:

$$z \in X \cap Y \triangleq z \in X \wedge z \in Y. \tag{1.3}$$

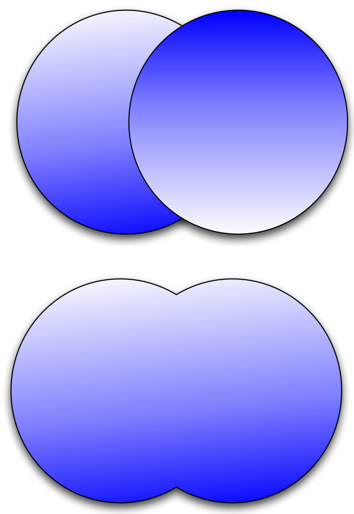

Figure 1.3: Set union

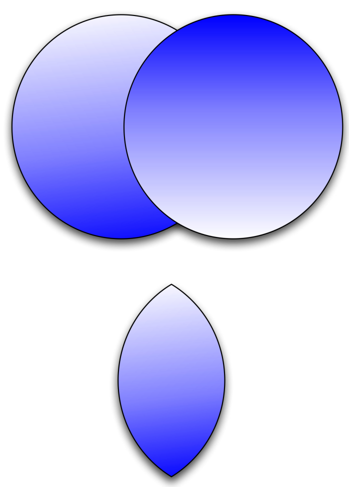

Figure 1.4: Set intersection

Finally, in set theory we can speak about set substraction:

$$z \in X \setminus Y \triangleq z \in X \wedge z \notin Y. \tag{1.4}$$

There are two sets that are important: empty set and universal set.

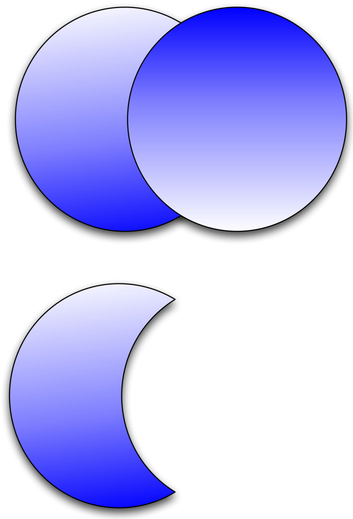

Figure 1.5: Set substraction

The empty set is empty, i.e. it does not contain any elements, but in order to define it in a proper way we need to use a different way:

$$x \in \emptyset \triangleq x \neq x. \tag{1.5}$$

This definition implies that nothing belongs to the empty set, but it also leads to a definition of the other important set, i.e., the universal set $\mathbb{U}$:

$$x \in \mathbb{U} \triangleq x = x. \tag{1.6}$$

The intuition that supports the idea of the universal set is simple: the empty set is such that everything belongs to it. But this intuition proved to be on the wrong tracks - see section 2.1 on page 31.

The notion of universal set allows us to define the set complement by means of set substration:

$$-X \triangleq \mathbb{U} \setminus X \tag{1.7}$$

Of course, one can define the complement in a different way:

$$y \in -X \triangleq \neg x \in X \tag{1.8}$$

If you know all these seven notions: $\in$, $\subseteq$, $\cap$, $\cup$, $\setminus$, $\emptyset$, and $\mathbb{U}$, you know the basics. Welcome to the primary school in set theory!

1.3.4 Beyond the basics

The seven notions enable us to spell out a number of laws of set theory. In fact, your ability to separate those laws from other formulas proves that you have a proper understanding thereof.

As usual, the number of laws in baby set theory is infinite, but below I listed the most important of them:

$$\emptyset = -\mathbb{U} \tag{1.9}$$
$$\mathbb{U} = -\emptyset \tag{1.10}$$
$$\emptyset \neq \mathbb{U} \tag{1.11}$$
$$\emptyset \subseteq X \tag{1.12}$$
$$X \subseteq \emptyset \to X = \emptyset \tag{1.13}$$
$$X \subseteq \mathbb{U} \tag{1.14}$$
$$\mathbb{U} \subseteq X \to X = \mathbb{U} \tag{1.15}$$
$$X \subseteq -X \to X = \emptyset \tag{1.16}$$
$$-X \subseteq X \to X = \mathbb{U} \tag{1.17}$$
$$X \subseteq X \tag{1.18}$$
$$X \subseteq Y \wedge Y \subseteq Z \to X \subseteq Z \tag{1.19}$$
$$X \cap Y = Y \cap Z \tag{1.20}$$
$$X \cap (Y \cap Z) = (X \cap Y) \cap Z \tag{1.21}$$
$$X \cap \emptyset = \emptyset \tag{1.22}$$
$$X \cap \mathbb{U} = X \tag{1.23}$$
$$X \cap Y \subseteq X \tag{1.24}$$
$$X \cup Y = Y \cup X \tag{1.25}$$
$$X \cup (Y \cup Z) = (X \cup Y) \cup Z \tag{1.26}$$
$$X \cup \emptyset = X \tag{1.27}$$
$$X \cup \mathbb{U} = \mathbb{U} \tag{1.28}$$
$$X \subseteq X \cup Y \tag{1.29}$$
$$X \cup (Y \cap Z) = (X \cap Y) \cup (X \cap Z) \tag{1.30}$$
$$X \cap (Y \cup Z) = (X \cup Y) \cap (X \cup Z) \tag{1.31}$$
$$X \cap Y \subseteq X \cup Y \tag{1.32}$$
$$-(X \cap Y) = -X \cup Y \tag{1.33}$$
$$-(X \cup Y) = -X \cap Y \tag{1.34}$$

Some of these laws are easy to prove, others are more troublesome. Let us start with the easy bit then!

There is a simple method that allows us to prove or disprove a number of laws/now-laws of our baby set theory. This method has four stages:

1. Is ϕ a law of set theory?

2. If $\phi = \Delta_1 = \Delta_2$, transform ϕ into $\forall\alpha\ [\alpha \in \Delta_1 \equiv \alpha \in \Delta_2]$!

3. Using the appropriate definitions remove from ϕ all symbols and phrases that are defined in set theory!

4. Remove all quantifiers!

5. Translate the resulting formula to the language of the propositional logic!

   (a) If Δ is neither the empty nor the universal set, translate "$\alpha \in \Delta$" as p!

   - when "Δ_1" $\neq$ "Δ_2", use different variables for your translations of "$\alpha \in \Delta_1$" and "$\beta \in \Delta_2$"!

   (b) Translate $\alpha \in \emptyset$ as $p \wedge \neg p$!

   (c) Translate $\alpha \in \mathbb{U}$ as $p \vee \neg p$!

6. Check whether the propositional formula is a law of logic!

   (a) If it is, then ϕ is a law of set theory.

   (b) If it is not, then ϕ is not a law of set theory.

Here are some examples of how to use this method:

$$\emptyset = -\mathbb{U}. \tag{1.35}$$

1. Is $\emptyset = -\mathbb{U}$ a law of set theory?

2. $\forall x\ [x \in \emptyset \equiv x \in -\mathbb{U}]$

3. $\forall x\ [x \in \emptyset \equiv \neg(x \in \mathbb{U})]$

4. $x \in \emptyset \equiv \neg(x \in \mathbb{U})$

5. $p \wedge \neg p \equiv \neg(p \vee \neg p)$

6. Since $p \wedge \neg p \equiv \neg(p \vee \neg p)$ is a law of propositional logic, $\emptyset = -\mathbb{U}$ is a law of set theory.

$$\emptyset \subseteq X \tag{1.36}$$

1. Is $\emptyset \subseteq X$ a law of set theory?

2. n/a

3. $\forall y\ (y \in \emptyset \rightarrow y \in X)$

4. $y \in \emptyset \rightarrow y \in X$

5. $p \wedge \neg p \rightarrow q$

6. Since $p \wedge \neg p \to q$ is a law of propositional logic, $\emptyset \subseteq X$ is a law of set theory.

$$X \subseteq X \tag{1.37}$$

1. Is $X \subseteq X$ a law of set theory?
2. n/a
3. $\forall y\ (y \in X \to y \in X)$
4. $y \in X \to y \in X$
5. $p \equiv p$
6. Since $p \equiv p$ is a law of propositional logic, $X \subseteq X$ is a law of set theory.

$$X \subseteq Y \wedge Y \subseteq Z \to X \subseteq Z. \tag{1.38}$$

1. Is $X \subseteq Y \wedge Y \subseteq Z \to X \subseteq Z$ a law of set theory?
2. n/a
3. $\forall v(v \in X \to v \in Y) \wedge \forall v(v \in Y \to v \in Z) \to \forall v(v \in X \to v \in Z)$
4. $(v \in X \to v \in Y) \wedge (v \in Y \to v \in Z) \to (v \in X \to v \in Z)$
5. $(p \to q) \wedge (q \to r) \to (p \to r)$
6. Since $(p \to q) \wedge (q \to r) \to (p \to r)$ is a law of propositional logic, $X \subseteq Y \wedge Y \subseteq Z \to X \subseteq Z$ is a law of set theory.

$$X \cup Y \subseteq X \cap Y \tag{1.39}$$

1. Is $X \cup Y \subseteq X \cap Y$ a law of set theory?
2. n/a
3. $\forall v(v \in X \vee v \in Y) \to \forall v(v \in X \wedge v \in Y)$
4. $(v \in X \vee v \in Y) \to (v \in X \wedge v \in Y)$
5. $p \vee q \to p \wedge q$
6. Since $p \vee q \to p \wedge q$ is *not* a law of propositional logic, $X \cup Y \subseteq X \cap Y$ is *note* a law of set theory.

However, some laws of baby set theory require more sophisticated methods, e.g. $\emptyset \neq \mathbb{U}$.

1.3.5 Families of sets, etc.

Set theory deals with all kinds of sets, for instance:

- set of all people
- set of fruits
- union of set of fruits and vegetables
- set of all electrons
- etc.

There is an interesting category of sets that do not contains such elements as people, fruits, and vegetables, but which contains sets themselves. Consider for instance, the set of two sets: empty and universal set. Or consider the following cases:

- set of all sets of plants
- set of all finite sets
- set of all sets
- ...

Beware that some of these sets are dangerous as they lead into inconsistency.

Set theory has a special tool for creating sets of sets (or *families* of sets). This tool is called the powerset. Consider the set of four cardinal directions: $\{w, e, s, n\}$, where

- w stands for west,
- e stands for east,
- s for south,
- and n for north.

You might be amazed to hear that this set has 16 subsets:

1. $\{w, e, s, n\}$
2. $\{e, s, n\}$
3. $\{w, s, n\}$
4. $\{w, e, n\}$
5. $\{w, e, s\}$
6. $\{w, e\}$

7. $\{w, n\}$
8. $\{w, s\}$
9. $\{e, s\}$
10. $\{e, n\}$
11. $\{s, n\}$
12. $\{w\}$
13. $\{e\}$
14. $\{s\}$
15. $\{n\}$
16. $\emptyset$

Among the subsets of $\{w, e, s, n\}$ there are some peculiarities:

- $\{w, e, s, n\}$ is its own subset
- $\emptyset$ is a subset of $\{w, e, s, n\}$
  - These two follow from definition 1.1 (page 8) - see laws 1.36 and 1.37 above.
- singletons:
  1. $\{w\}$
  2. $\{e\}$
  3. $\{s\}$
  4. $\{n\}$

  - These are kind of borderline sets that include exactly one member each. But if a set that contains two or three objects is all right, then the set that contains only one item should be fine as well.

If "X" denotes a set, then "$\wp(X)$" or "2^X" will denote the powerset of X. The latter notation is understandable given the fact that if X has n elements, then the powerset of X, i.e., 2^X has 2^n elements.

$$Y \in \wp(X) \triangleq Y \subseteq X. \tag{1.40}$$

The notion of powerset, or rather the operation of powerset creation, leads from set to sets of sets. In set theory there is an opposite notion, or operation, namely, the notion of (universal) union:

$$y \in \bigcup X \triangleq \exists Z \in X \; y \in Z \tag{1.41}$$

In order to understand how this definition works consider first a simple case where $X = \{A, B\}$. Then it follows from 1.41 that

$$y \in \bigcup\{A, B\} \triangleq \exists Z \in \{A, B\}\ y \in Z,$$

and this entails that

$$y \in \bigcup\{A, B\} \triangleq y \in A \vee y \in B,$$

which is equivalent to definition 1.2.

Let's now have some fun and play with the two above notions: powerset and union. We start from set $\{w, e, s, n\}$. Then we create its powerset

$\wp(\{w,e,s,n\}) = \{\{w,e,s,n\},\{e,s,n\},\{w,s,n\},\{w,e,n\},\{w,e,s\},\{w,e\},\{w,n\},\{w,s\},\{e,s\},\{e,n\},\{s,n\},\{w\},\{e\},\{s\},\{n\},\emptyset\}$

And now we will create the union of the powerset, i.e.

$$\bigcup \wp(\{w,e,s,n\}) = \{w,e,s,n\}.$$

Consider also the following examples:

$$\bigcup\{\{w,e,s,n\},\{e,s,n\},\{w,s,n\},\{w,e,n\},\{w,e,s\}\} = \{w,e,s,n\}$$

$$\bigcup\{\{w\},\{e\},\{s\},\{n\}\} = \{w,e,s,n\}$$

$$\bigcup\{\{w\},\{w,e\},\{e\},\emptyset\} = \{w,e\}$$

$$\bigcup\emptyset = \emptyset$$

In sum, the union of a family of sets is the set of all elements of all elements of this family or the set of all elements of those sets.

Since definition 1.41 is a special, borderline case of definition 1.2, we could also try to generalise the notion of product (definition 1.3):

$$y \in \bigcap X \triangleq \forall Z \in X \ y \in Z. \tag{1.42}$$

Again first note definition 1.3 is a special case of 1.42:

$$y \in \bigcap \{A, B\} \triangleq \forall Z \in \{A, B\} \ y \in Z$$

$$y \in \bigcap \{A, B\} \triangleq y \in A \land y \in B.$$

Let's again have some fun and play with the notion of product.

$\wp(\{w,e,s,n\}) = \{\{w,e,s,n\},\{e,s,n\},\{w,s,n\},\{w,e,n\},\{w,e,s\},\{w,e\},\{w,n\},\{w,s\},\{e,s\},\{e,n\},\{s,n\},\{w\},\{e\},\{s\},\{n\},\emptyset\}$

And now we will create the union of the powerset, i.e.

$$\bigcap \wp(\{w,e,s,n\}) = \emptyset.$$

Consider also the following examples:

$$\bigcap\{\{w,e,s,n\},\{w,s,n\},\{w,e,n\},\{w,e,s\}\} = \{w\}$$

$$\bigcap\{\{w,e,s,n\},\{w,e,s\},\{w,e,n\}\} = \{w,e\}$$

$$\bigcap\{\emptyset\} = \emptyset$$

In sum, the product of a family of sets is the set of all such elements that belong to each element of this family or the set of common elements among those sets.

The generalised notions of union and product are interrelated:

$$X \neq \emptyset \to \bigcap X \subseteq \bigcup X. \tag{1.43}$$

In order to show this we need to go far beyond the simple method described in section 1.3.4 on page 12. Namely, we need to employ the full strength of the natural deduction method for predicate logic. We will prove first that for any x it is the case that

$$X \neq \emptyset \to (x \in \bigcap X \to x \in \bigcup X)$$

and than using a rule derived from rule $\mathbb{I}_\forall$ we get

$$X \neq \emptyset \to \forall x(x \in \bigcap X \to x \in \bigcup X),$$

which is equivalent to 1.43.

1.	$X \neq \emptyset$	*premise*
2.	$x \in \bigcap X$	*premise*
3.	$x \in \bigcap X \triangleq \forall Y \in X\ x \in Y$	$def.$ 1.42 : $\mathbb{E}_\wedge : 2$
4.	$(x \in \bigcap X \triangleq \forall Y \in X\ x \in Y) \wedge x \in \bigcap X \to \forall Y \in X\ x \in Y$	*law* : $(p \equiv q) \wedge p \to q$
5.	$\forall Y \in X\ x \in Y$	$\mathbb{E}_\to : 4, (\mathbb{I}_\wedge : 3, 2)$
6.	$\forall Y(Y \in X \to x \in Y)$	$def.$
7.	$X \neq \emptyset \to \exists Y\ Y \in X$	*thesis*
8.	$\exists Y\ Y \in X$	$\mathbb{E}_\to : 7, 1$
9.	$Y_0 \in X$	$\mathbb{E}_\exists : 8$
10.	$Y_0 \in X \to x \in Y_0$	$\mathbb{E}_\forall : 6$
11.	$x \in Y_0$	$\mathbb{E}_\to : 10$
12.	$Y_0 \in X \wedge x \in Y_0$	$\mathbb{I}_\wedge : 9, 11$
13.	$\exists Y(Y \in X \wedge x \in Y)$	$\mathbb{I}_\exists : 12$
14.	$\exists Y \in Xx \in Y$	$def. : 13$
15.	$x \in \bigcup X \triangleq \exists Y \in X\ x \in Y$	$def.$ 1.41
16.	$(x \in \bigcup X \triangleq \exists Y \in X\ x \in Y) \wedge \exists Y \in Xx \in Y \to x \in \bigcup X$	*law* $(p \equiv q) \wedge q \to p$
	$x \in \bigcup X$	$\mathbb{E}_\to : 16, (\mathbb{I}_\wedge : 15, 14)$

However, in order to complete this proof we need to prove the thesis we used, i.e.,

$$X \neq \emptyset \to \exists Y\ Y \in X.$$

Unfortunately, this proof requires more sophisticated machinery that will be introduced in section 3.1. Namely, we need the axiom of extensionality for sets:

$$\forall z\ (z \in X \equiv z \in Y) \to X = Y. \tag{Extensionality}$$

Given the deductive strength of axiom Extensionality we are in a position to prove that

$$X \neq \emptyset \to \exists y\ y \in X. \tag{1.44}$$

Proof. Assume otherwise, i.e., assume that X is not identical to the empty set, still it has no elements. Axiom Extensionality implies that there is x_0 such that either

$$x_0 \in X \wedge x_0 \notin \emptyset$$

or

$$x_0 \notin X \wedge x_0 \in \emptyset.$$

The first case is excluded by the *reductio* assumption. And the second would imply that $x_0 \in \emptyset$. Given definition 1.5 we would then get $x_0 \neq x_0$, which is inconsistent with one of thesis of predicate logic. □

Oddly enough, the following theorem holds:

$$\bigcap \emptyset = \mathbb{U}. \tag{1.45}$$

1.3.6 Relations

Oddly enough, it turns out that using sets we are able to define relations.

The route from sets to relations leads through the notion of ordered tuple.

Note that finite sets themselves may be seen as tuples:

- sets with two elements as couples, e.g.,

$$\{John, Ann\}$$

- sets with three elements as triples, e.g.,

$$\{John, Ann, Eva\}$$

- sets with four elements as quadruples, e.g.,

$$\{John, Ann, Eva, Peter\}$$

- etc.

Note however that these are unordered tuples, i.e., the ordering between their elements do not matter:

$$\{John, Ann\} = \{Ann, John\}.$$

In order to get ordered tuples, i.e. those for which the ordering does matter, either we need to introduce another primitive notion to set theory or use the following definition.

$$< x, y > \triangleq \{\{x\}, \{x, y\}\}. \tag{1.46}$$

Of course, "$< x, y >$" denotes a couple whose first member is x and second member is y.

Definition 1.46 adequately grasps our understanding of couples as ordered pairs since it implies the following theorem:

$$< x_1, y_1 > = < x_2, y_2 > \equiv x_1 = x_2 \wedge y_1 = y_2. \tag{1.47}$$

Proof. $\rightarrow$
Assume that

$$< x_1, y_1 >=< x_2, y_2 > .$$

Then definition 1.46 implies that

$$\{\{x_1\}, \{x_1, y_1\}\} = \{\{x_2\}, \{x_2, y_2\}\}.$$

Then

1. $x_1 = x_2$ or $x_1 = x_2 = y_2$
2. $x_1 = y_1 = x_2$ or $\{x_1, y_1\} = \{x_2, y_2\}$

but

1. if $x_1 = x_2$ then $\{x_1, y_1\} = \{x_2, y_2\}$
2. if $x_1 = x_2 = y_2$ then $x_1 = y_1 = x_2$

Consequently,

$$x_1 = x_2 \wedge y_1 = y_2.$$

$\leftarrow$
If $x_1 = x_2$ and $y_1 = y_2$ then

$$\{\{x_1\}, \{x_1, y_1\}\} = \{\{x_2\}, \{x_2, y_2\}\}.$$

And definition 1.46 implies that

$$< x_1, y_1 >=< x_2, y_2 > .$$

$\square$

Definition 1.46 mentions only ordered couples. What about ordered triples, quadruples, etc.? Well, they are now piece of cake:

$$< x_1, x_2, \ldots, x_n > \triangleq < x_1, < x_2, \ldots, x_n >> . \tag{1.48}$$

For instance, triples can be obtained as couples of couples:

$$< x_1, x_2, x_3 > \triangleq < x_1, < x_2, x_3 >> . \tag{1.49}$$

Now since we know what ordered tuples are, we can say that relations are sets of ordered tuples:

- relation of fatherhood is the set of ordered pairs such that its first element is the father of the second element
- relation of hatred is the set of ordered pairs such that its first element hates the second element

- relation of being greater than (i.e., $>$) is is the set of ordered pairs such that its first element is greater than the second element
- etc.

Formally, the path to relations leads through the gate of the Cartesian product. Intuitively, the Cartesian product $X \times Y$ of two sets X and Y is the set of all couples such that their first elements always comes from X and their second elements always comes from Y:

$$< x, y > \in X \times Y \triangleq x \in X \wedge y \in Y. \tag{1.50}$$

Then a (binary) *relation* in set X is any subset of the Cartesian product $X \times X$, i.e. it is a set of ordered couples. In general, a *relation* is a set of ordered tuples, however, in what follows we will speak mainly about binary relations.

If R is a relation, then the set of all first (i.e., left-hand side) arguments of R is called the *domain* of R. Similarly, the set of all second (i.e., right-hand side) arguments of R is called the *range* of R.

$$x \in \mathbb{DOM}(R) \triangleq \exists y \; R(x, y). \tag{1.51}$$

$$x \in \mathbb{RANGE}(R) \triangleq \exists y \; R(y, x). \tag{1.52}$$

Examples:

1. relation of fatherhood:
   (a) domain: set of fathers
   (b) range: set of children
2. relation of begin greater than:
   (a) domain: set of numbers
   (b) range: set of numbers

Set theory not only makes it possible to speak about relations, but also provides with means to classify them or describe their formal properties.

Definition 1. *Relation R is* reflexive *(in set X) if and only if for all $x \in X$, $R(x, x)$.*

Definition 2. *Relation R is* irreflexive *(in set X) if and only if for all $x \in X$, $\neg R(x, x)$.*

Definition 3. *Relation R is* symmetric *(in set X) if and only if for all $x, y \in X$, if $R(x, y)$, then $R(y, x)$.*

Definition 4. *Relation R is* asymmetric *(in set X) if and only if for all $x, y \in X$, if $R(x, y)$, then $\neg R(y, x)$.*

Definition 5. *Relation R is* antisymmetric *(in set X) if and only if for all $x, y \in X$, if $R(x, y)$ and $R(y, x)$, then $x = y$.*

Definition 6. *Relation R is* transitive *(in set X) if and only if for all $x, y, z \in X$, if $R(x, y)$ and $R(y, z)$, then $R(x, z)$.*

Definition 7. *Relation R is* intransitive *(in set X) if and only if for all $x, y, z \in X$, if $R(x, y)$ and $R(y, z)$, then $\neg R(x, z)$.*

Definition 8. *Relation R is* connective *(in set X) if and only if for all $x, y \in X$, either $R(x, y)$ or $R(y, x)$ or $x = y$.*

Table 1.1 collects some examples of relations classified with respect to those formal properties.

Relation	reflexivity	symmetry	transitivity
being a father of	irreflexive	asymmetric	intransitive
being a compatriot of	reflexive	symmetric	transitive
being similar to	reflexive	symmetric	neither transitive nor intransitive
love	neither reflexive nor irreflexive	neither symmetric nor asymmetric nor antisymmetric	neither transitive nor intransitive
divides	reflexive	antisymmetric	transitive
being parallel to	reflexive	symmetric	transitive
$>$	irreflexive	asymmetric	transitive

Table 1.1: Formal properties of relations - examples

In set theory theory there are two types of relations that are of special interest: orderings, equivalence relations, and functions.

An *ordering* (in set X) is any relation that is reflexive, antisymmetric, and transitive (in X). When a relation is asymmetric and transitive, then it is sometimes called *strict ordering*. An ordering R is *linear* if it is connective.

An *equivalence relation* (in set X) is any relation that is reflexive, symmetric, and transitive (in X).

Equivalence relations are of uttermost importance in science. Briefly speaking, each equivalence relation defines an (abstract) property/quality. The role of equivalence relation is highlighted by the so-called abstraction theorem.

Let R be the equivalence relation that relates material objects of the same weight, i.e., $R(x, y)$ means that x weighs the same as y. Pick up one material object, let it be John, and consider the following set

$$\{x : R(x, \text{John})\}.$$

Our set contains all material objects whose weight it equal to John's. Assume that John weighs 75 kilograms. Then our set will contain all objects that weighs 75 kilograms. Now pick up another material object, let it be John's cat, and consider the following set

$$\{x : R(x, \text{John's cat})\}.$$

Our set contains all material objects whose weight it equal to John cat's. Assume that John's cat weighs 5 kilograms. Then our set will contain all objects that weighs 5 kilograms. For each object we can reiterate the whole procedure and in the end we will get a set of sets whose members are bound to have the same value. That is to say, there will be a set of objects weighing 1 kliogram, a set of objects weighing 2 kilogram, etc. Note that

1. none of these sets will be empty;

2. no two of them will intersect;

3. the union of all of them will cover the whole set of material objects.

Thus, one can claim that our relation defines the notion of weight. The weight of 1 kilogram is defined by the set of all objects weighing 1 kilogram. The weight of 2 kilogram is defined by the set of all objects weighing 2 kilogram. And so on. This piece of reasoning exemplifies an important theorem in set theory.

Theorem 1. *The quotient set of set X by equivalence relation R is a partition of X.*

I will not prove this theorem, but let me explain at least the notions occurring therein.

Assume that R is an equivalence relation in set X. If $y \in X$, then each set defined as follows

$$\{y \in X : R(y, x)\}$$

is called an *equivalence class* of relation R defined by y. The set of equivalence classes defined by all elements of X is called the *quotient set* of X by equivalence relation R. Note that each quotient set is a family, i.e., it is a set whose members are sets.

Now a family X of sets is a *partition* of set Y provided that

1. no element of X is empty;
2. if Z_1 and Z_2 belong to X, then $Z_1 \cap Z_2 = \emptyset$;
3. $\bigcup X = Y$.

A *function* (in set X) is any relation R that satisfies the following condition:

$$\forall x, y_1, y_2 \in X[R(x, y_1) \wedge R(x, y_2) \rightarrow y_1 = y_2].$$

If R is a function and $R(x, y)$, then we usually write $y = f(x)$ instead of $R(x, y)$, i.e., we use functional symbol "f" instead of predicate "R".

Consider now some examples:

1. examples of relations being functions:
   - x was born on y
   - x is the capital (city) of y
   - $y = x^2$
2. examples of non-functions:
   - x is the mother of y
   - x is a city in y
     - because Slobomir is a city in two countries:
       (a) Bosnia and Herzegowina
       (b) Serbia
   - $x^2 + y^2 = 1$

When relation R has $n+1$ arguments, one can speak about n-ary functions:

$$\forall x_1, x_2, \ldots, x_n, y_1, y_2 \in X[R(x_1, x_2, \ldots, x_n, y_1) \wedge R(x_1, x_2, \ldots, x_n, y_2) \rightarrow y_1 = y_2].$$

Then, of course, $y = f(x_1, x_2, \ldots, x_n)$.

Any function whose domain and range are subsets of the set of real numbers can be depicted in the Cartesian coordinate system as a function graph - see figure 1.6.

Assume that f is a unary function (i.e., binary relation). Then set

$$f(X) \triangleq \{y : \exists x \in X\ f(x) = y\}$$

is called the *image* of X under function f.

If f is a function with domain X and range Y and $Y \subseteq Z$, then we say that

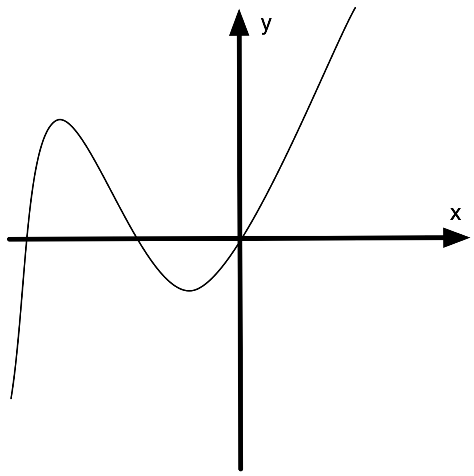

Figure 1.6: A graph of a binary function

1. f maps X onto Y - written: $f : X \underset{\text{on}}{\rightarrow} Y$;
2. f maps X into Z - written: $f : X \rightarrow Z$.

In set theory we will also use a special type of functions, called injections.

Definition 9. *A function f is* injective *if and only if*

$$\forall x_1, x_2\ [f(x_1) = f(x_2) \rightarrow x_1 = x_2].$$

For example, function $y = x + 2$ is injective, but function $y = x^2$ is not because $2^2 = (-2)^2$.

If an injective function f maps X onto Y, we say that f is called a *bijection* from X onto Y.

When function f is bijective, we can define the notion of inverse function. Assume that f is a bijection from X onto Y. Then function f^{-1} defined as below is called the *inverse* function to f:

$$y = f^{-1}(x) \triangleq x = f(y).$$

Obviously, f^{-1} is bijective as well.

For instance, consider function f_{CF} that converts a temperature in degrees Celsius to degrees Fahrenheit:

$$f_{CF}(x) = \frac{9}{5}x + 32.$$

Since f_{CF} is a bijection from the set of the degrees Celsius to the set of degrees Fahrenheit, we can define its inverse:

$$(f_{CF})^{-1}(x) = \frac{5}{9}(x - 32).$$

Function $(f_{CF})^{-1}$ converts a temperature in degrees Fahrenheit to degrees Celsius.

Chapter 2

When things go wrong in logic . . .

2.1 Set theory dies . . .

Philosophers (and some other strange folks) have invented over the centuries a number of logical puzzles, which are likely to confuse anybody. One of the most famous is the Liar Paradox. Assume that someone say

$$\text{I am lying.} \tag{2.1}$$

Note that

1. if 2.1 it true, then 2.1 is false
2. if 2.1 it false, then 2.1 is true

But perhaps it would easier to consider the following sentence:

$$\text{Sentence 2.2 is not true.} \tag{2.2}$$

One can prove that:

$$\text{Sentence 2.2 is not true iff sentence 2.2 is true.} \tag{2.3}$$

1. If sentence 2.2 is not true, then sentence 2.2 is true.
   (a) Assume that sentence 2.2 is not true.
   (b) It is not true that sentence 2.2 is not true.
   (c) Sentence 2.2 is not true.
2. If sentence 2.2 is true, then sentence 2.2 is not true.
   (a) Assume that sentence 2.2 is true.

(b) Sentence 2.2 is not true.

However, neither of these reasoning belongs to set theory since they do not involve the notion of set. Consider then the following piece of reasoning.

There exists a lot of sets - see in particular sections 1.3.2, 1.3.5, and 1.3.6. Although some of these sets are strange or difficult to grasp, they are normal in this sense that do not contain themselves.

- set of all human beings contains human beings and but not sets
- set of all natural numbers contains natural number but not sets [1]
- powerset of all human beings contains subsets of set of human beings, so it does contain sets, but it does not contain itself
- ...

In sum, if A is a set, then for each such case, i.e., for each normal set, it is false that $A \in A$. It would be very strange indeed if a set contained itself.

Still, if you think about some "big" or abstract sets, you will find some examples of abnormal sets:

- set of all infinite sets is infinite, so it contains itself
- set of all sets contains all sets, so it contains itself
-

Thus, we can draw a line between normal sets, for which $A \notin A$, and abnormal sets, for which $A \in A$.

Let us focus not on the good guys, i.e., let us focus on the set of all normal sets:

$$A \in \mathbb{R} \triangleq A \notin A. \tag{2.4}$$

This definition looks innocent, does it not? But its consequences are disastrous.

1.	$\mathbb{R} \in \mathbb{R} \equiv \mathbb{R} \notin \mathbb{R}$	substitution in 2.4
2.	$(p \equiv \neg p) \to p \wedge \neg p$	law of propositional logic
3.	$(\mathbb{R} \in \mathbb{R} \equiv \mathbb{R} \notin \mathbb{R}) \to \mathbb{R} \in \mathbb{R} \wedge \mathbb{R} \notin \mathbb{R}$	substitution in 2
4.	$\mathbb{R} \in \mathbb{R} \wedge \mathbb{R} \notin \mathbb{R}$	$\mathbb{E}_{\to} : 3, 1$
5.	$\mathbb{R} \in \mathbb{R}$	$\mathbb{E}_{\wedge} : 4$
6.	$\mathbb{R} \notin \mathbb{R}$	$\mathbb{E}_{\wedge} : 4$

... we got an inconsistency, so we are dead (as logicians).

Except for finishing off us nicely, this reasoning also shows that there are two inconsistent answers to the question "Is the set of all normal sets (i.e., $\mathbb{R}$) normal?": yes and no.

The above problem is called "Russell's paradox". There is a similar paradox, which is less serious and, for that reason, more student friendly. It is called "Barber paradox".

[1] At least it seems so, but be ready for a nasty surprise in section 3.4!

> Suppose there is a town with just one male barber; and that every man in the town keeps himself clean-shaven: some by shaving themselves, some by attending the barber. It seems reasonable to imagine that the barber obeys the following rule: He shaves all and only those men in town who do not shave themselves. Under this scenario, we can ask the following question: Does the barber shave himself? Asking this, however, we discover that the situation presented is in fact impossible:
>
> - If the barber does not shave himself, he must abide by the rule and shave himself.
> - If he does shave himself, according to the rule he will not shave himself.
>
> (from `http://en.wikipedia.org/wiki/Barber_paradox`)

In sum,

$$\text{Barber shaves himself iff barber does not shave himself.} \tag{2.5}$$

What shall we do to this barber? And more importantly, what shall we do about set theory?

Wrapping up note that all three problems mentioned here are based on the same principle:

$$p \equiv \neg p \tag{2.6}$$

where:

1. $p =$ Sentence 2.2 is true,
2. $p = \mathbb{R} \in \mathbb{R}$,
3. $p =$ Barber shaves himself.

2.2 ...and it rises from the dead.

In order to avoid the Russel antinomy we need to identify its roots. Generally speaking, the source of this, and other antinomies, is a certain indulgence in set-theoretic creativity. This tolerance allows one to "have" any set that one wants: if you want the set of all sets here it is. If you need the set of all normal sets, no problem. There is a formal pattern that makes this creativity possible (justifies it). This patter is called the axiom of comprehension:

$$\exists X \forall y\ [y \in X \equiv \phi(y)], \tag{2.7}$$

where $\phi(y)$ represents a sentential frame with y as a free variable.

In less formal terms you can spell out this axiom saying that for any condition there exists a set whose all members (and only those members) satisfy this condition. Note that if you pick up a contradictory condition, you will end up defining the empty set:

1. $\exists X \forall y\ [y \in X \equiv y \neq y]$ substitution in 2.7
2. $y \in \emptyset \equiv y \notin y$ 1

In a similar way one can get from 2.7 to 2.4:

1. $\exists X \forall y\ [y \in X \equiv y \notin y]$ substitution in 2.7
2. $y \in \mathbb{R} \equiv y \neq y$ 1

Thus, the source of the trouble with sets is the axiom of comprehension. Consequently, all solutions suggest replacing this axiom with some other, less disastrous, principle. Usually these solutions are classified in the following way:

1. Change the language of set theory!
   (a) simple theory of types
   (b) ramified theory of types
2. Modify the axiom of comprehension
   (a) Zermelo-Fraenkel set theory
   (b) Neumann-Bernays-Godel set theory
   (c) ...

The remaining part of this course will follow Zermelo-Frankel set theory. Here I will briefly explain the main ideas of other solutions:

Simple theory of types This solution suggests that we should modify the language of set theory in such a way that we could not formulate 2.4 any more. The modification of the language is based on the following ontological picture. The universum is divided into levels:

Level ... ...

Level 2 set of sets of individuals

Level 1 set of individuals

Level o individuals

The simple theory of types requires then that formula

$$x \in A$$

is meaningful if and only if A belongs to the level just above the level of x. So one can say that John belongs to the set of human beings because John is to be found at level 0 and the set of human being belongs level 1. But one cannot say that $x \in x$ irrespective of the level to which x belongs. Consequently, one cannot also say anything that resembles 2.4.

Neumann-Bernays-Godel set theory This theory introduces the distinction between sets and (proper) classes. If for some B, $A \in B$, then A is a set; otherwise it is a (proper) class. Then we can modify the axiom of comprehension as follows:

$$\exists X \forall y\ [y \in X \equiv y \text{ is a set} \wedge \phi(y)]. \tag{2.8}$$

Given 2.8 we will not get any inconsistency, but only the thesis that $\mathbb{R}$ is a proper class.

1.	$\exists X \forall y\ [y \in X \equiv y \text{ is a set} \wedge y \notin y]$	substitution in 2.8
2.	$y \in \mathbb{R} \equiv y \text{ is a set} \wedge y \notin y$	1
3.	$\mathbb{R} \in \mathbb{R} \equiv \mathbb{R} \text{ is a set} \wedge \mathbb{R} \notin \mathbb{R}$	2
4.	$\neg\mathbb{R}$ is a set	$3, (p \equiv q \wedge \neg p) \rightarrow \neg q$
5.	$\mathbb{R}$ is a proper class	4

Chapter 3

Zermelo-Fraenkel set theory

Zermelo-Fraenkel set theory (also ZF theory) is a solution to the set-theoretic antinomies. The solution consists in a number of axioms that specify which sets exist.

3.1 Axioms

One axiom we already know (see Extensionality on page 21):

$$\forall x\ (x \in A \equiv x \in B) \to A = B.$$

Instead of (NOT: besides) axiom of comprehension ZF theory has the axiom of subsets. If $\phi(x)$ does not contain variable A, but does contain free variable x:

$$\exists A\ \forall x\ [x \in A \equiv x \in B \wedge \phi(x)]. \qquad \text{(Subsets)}$$

Sometimes one needs a stronger version of Subsets:

$$\begin{gathered}\forall x, y_1, y_2\ [\phi(x, y_1) \wedge \phi(x, y_2) \to y_1 = y_2] \to \\ \to \forall B\ \exists A\ \forall y\ [y \in A \equiv \exists x \in B\ \phi(x, y)].\end{gathered} \qquad (3.1)$$

where: $\phi(x, y)$ contains two free variables x and y, but does not contain variable A. Axiom 3.1 is called the axiom of replacement. Note that Subsets follows from 3.1:

1.	$\phi(x, y) \triangleq x = y \wedge \varphi(x)$	definition
2.	$\forall x, y_1, y_2\ [x = y_1 \wedge \varphi(x) \wedge x = y_2 \wedge \varphi(x) \to y_1 = y_2]$	law of logic
3.	$\forall B\ \exists A\ \forall y\ [y \in A \equiv \exists x \in B\ (x = y \wedge \varphi(x))]$	$\mathbb{E}_{\to}$: substitution in $3.1, 2$
4.	$\forall B\ \exists A\ \forall y\ [y \in A \equiv y \in B \wedge \varphi(y))]$	

Other axioms:

$$\exists A\ \forall x\ [x \in A \equiv x = y_1 \vee x = y_2]. \qquad \text{(Pair)}$$

$$\exists A\ \forall x\ [x \in A \equiv \exists y \in B\ x \in y]. \qquad \text{(Sum)}$$

$$\exists A\ \forall x\ [x \in A \equiv x \subseteq B]. \tag{Powerset}$$

$$\exists A\ [\emptyset \in A \wedge \forall x \in A\ x \cup \{x\} \in A]. \tag{Infinity}$$

$$\forall x \in B\ x \neq \emptyset \wedge \forall y_1, y_2 \in B\ [y_1 \neq y_2 \rightarrow y_1 \cap y_2 = \emptyset] \rightarrow \exists A\ \forall x \in B\ \exists y\ A \cap x = \{y\}. \tag{Choice}$$

One can classify these axioms into several groups:

1. existential (explosive) axioms
   (a) absolute
      i. Pair
      ii. Infinity
   (b) relative
      i. Subsets or 3.1
      ii. Sum
      iii. Powerset
      iv. Choice
2. non-existential (implosive) axioms
   (a) Extensionality

3.2 What about antinomies in ZF theory?

The axiom of comprehension (2.7) paved a way towards definition 2.4, which is inconsistent. ZF theory has the axioms of subsets (Subsets) instead of 2.7, so what happened with definition 2.4 there?

1.	$\exists A\ \forall x\ [x \in A \equiv x \in B \wedge x \notin x)]$	substitution in Subsets
2.	$x \in \mathbb{R}_B \equiv x \in B \wedge x \notin x$	1

So instead of definition 2.4 you get something like line 2 above. Note that this equivalence is not even a definition of a single concept. And it is not dangerous as it does not lead into inconsistency:

3.	$\mathbb{R}_B \in \mathbb{R}_B \equiv \mathbb{R}_B \in B \wedge \mathbb{R}_B \notin \mathbb{R}_B$	2
4.	$\mathbb{R}_B \notin B$	$3, (p \equiv q \wedge \neg p) \rightarrow \neg q$
5.	$\mathbb{R}_B \notin \mathbb{R}_B$	$3, (p \equiv q \wedge \neg p) \rightarrow \neg p$

One can show that also other antinomies dissolve in ZF theory.

3.3 Back to the basics

The axioms of ZF theory should constitute a (conceptual) foundation for any notion in set theory. Then what about the notions that we started our journey with?

1. union of (two) sets (definition 1.2)

1.	$\exists A\ \forall x[x \in A \equiv x = X \vee x = Y]$	substitution in Pair
2.	$x \in \{X, Y\} \equiv x = X \vee x = Y$	1
3.	$\exists A\ \forall x\ [x \in A \equiv \exists y \in \{X, Y\}\ x \in y]$	substitution in Sum
4.	$\exists A\ \forall x\ [x \in A \equiv x \in X \vee x \in Y]$	3
5.	$x \in X \cup Y \equiv x \in X \vee x \in Y$	4

2. product of (two) sets (definition 1.3)

1.	$\exists A\ \forall x\ [x \in A \equiv x \in X \cup Y \wedge (x \in X \wedge x \in Y)]$	substitution in Subsets
2.	$\exists A\ \forall x\ [x \in A \equiv (x \in X \vee x \in Y) \wedge (x \in X \wedge x \in Y)]$	definition1.2
3.	$x \in X \cap Y \equiv (x \in X \vee x \in Y) \wedge (x \in X \wedge x \in Y)$	2
4.	$x \in X \cap Y \equiv x \in X \wedge x \in Y$	$2, (p \vee q) \wedge (p \wedge q) \equiv p \wedge q$

3. set-theoretic difference (definition 1.4)

1.	$\exists A\ \forall x\ [x \in A \equiv x \in X \wedge x \notin Y]$	substitution in Subsets
2.	$x \in X \setminus Y \equiv x \in X \wedge x \notin Y$	1

4. empty set (definition 1.5)

1.	$\exists A\ \forall x\ [x \in A \equiv x \in X \wedge x \neq x]$	substitution in Subsets
2.	$x \in X \wedge x \neq x \equiv x \neq x$	law of logic
3.	$\exists A\ \forall x\ [x \in A \equiv x \neq x]$	$1, 2$
4.	$x \in \emptyset \equiv x \neq x$	3

Still, definition of universal set (1.6) and definition of set complement (1.7) are not derivable in ZF theory since they would lead to inconsistency.

3.4 Cardinal numbers

3.4.1 Let's count!

The axioms of ZF theory specify what sets are. Now it is time to count those sets or rather to count their members.

How many hedgehogs are there is my garden or how many stars are out there? It is relatively easy to answer such questions provided that one is focused and has a lot of time (in the latter case). It is little more difficult to explain what counting is or what is means that there are four hedgehogs in my garden. When one learns in a primary school how to count, one may use fingers to count. Surprisingly enough, set theory starts from this very idea of "counting by fingers".

First, set theory defines the notion of equinumerosity.

Definition 10. *Set A is* equinumerous *to set B (written: $A \sim B$) if and only if there exists a bijective function (mapping) from A onto B.*

The bijective function that definition 10 postulates is to guarantee that there is a one-to-one correspondence between all members of A and B. So if you want to know whether one set has exactly the same number of elements as another, you need to find such mapping. When we deal with finite sets, this definition seems to be an overshot as it is obvious what it means that two sets have the same number of elements. The things get much more complicated when we reach the infinite (or the infinite*s*).

Note that relation $\sim$ is reflexive, symmetric, and transitive, i.e., it is an equivalence relation. One may be then tempted (and some were actually tempted) do define the notion of number as an quotient set. For instance, one may be tempted to say that 2 is the set of all sets that are equinumerous to the set of my hands. One must resist this temptation, however, because it would lead to the antinomy of the set of all cardinal numbers.

Although one may define the notion of cardinal number in ZF theory, the proper definition requires a number of notions that will be introduced later on in this lecture. Thus, for the time being we will treat it as a primitive, i.e., undefined, term. If A is a set, then "$|A|$" will denote the cardinal number of A, i.e., this property of A that specifies how many elements there are in A.[1]

Being undefined does not mean not having any properties. For the time being the following properties will be necessary:

$$|A| = |B| \equiv A \sim B. \tag{3.2}$$

$$\forall A\ \exists \mathfrak{n}\ |A| = \mathfrak{n}. \tag{3.3}$$

$$\forall\ \mathfrak{n}\ \exists A\ \mathfrak{n} = |A|. \tag{3.4}$$

3.4.2 Beyond the finite

It is a usual custom to identify cardinal numbers of finite sets with natural numbers:

- $|\emptyset| = 0$
- $|\{\emptyset\}| = 1$
- $|\{\emptyset, \{\emptyset\}\}| = 2$
- etc.

What about the set of *all* natural numbers $\mathbb{N}$? Since there exists no greatest natural number, no natural number $n \in \mathbb{N}$ is such that $|\mathbb{N}| = n$. In fact the case

[1] I said "cardinal number" and not simply "number" because in set theory one also defines the notion of ordinal numbers. The simplest explanation of this difference points out to the difference between cardinal numerals: one, two, three, etc. and ordinal numerals: first, second, third, etc.

of $\mathbb{N}$ is our first encounter with infinity. To capture the cardinalities of infinite sets, set theory introduces infinite numbers.

$$|\mathbb{N}| \triangleq \aleph_0. \tag{3.5}$$

Thus, $\aleph_0$ is an infinite (cardinal) number that specifies how many natural numbers there are.

If there exists a natural number n such that $|A| = n$, then A is called finite; otherwise, it is infinite. If an infinite set A is of cardinality $\aleph_0$, then it is called *countable*; otherwise it is *uncountable*. These definitions imply the following hierarchy:

1. finite sets
2. infinite sets
   (a) countable sets
   (b) uncountable sets

Are there any uncountable sets? Before we answer this question let us first ponder over some more examples of countable sets.

First, by definition the set of all natural numbers is countable.

Secondly, note that set A is countable if and only if one can arrange all of its members into an infinite series without repetitions. The reason for this fact is that if set A is countable then and only then there exists a bijective mapping between A and $\mathbb{N}$. And this mapping in effect arrange all elements of A in an infinite series. This fact is quite useful since it facilitates our search for countable and uncountable sets.

Fact 1. *The set of all even numbers is countable.*

Proof. Since we can arrange all even numbers in a infinite series:

$$2, 4, 6, \ldots, 2 * n, \ldots$$

the set of all even numbers is of cardinality $\aleph_0$, i.e., is countable. □

Fact 2. *The set of all prime numbers is countable.*

Proof. Since we can arrange all even numbers in a infinite series:

$$1, 2, 3, 5, 7, 9, \ldots$$

the set of all prime numbers is of cardinality $\aleph_0$, i.e., is countable. □

Fact 3. *The set of all integers is countable.*

Proof. Since we can arrange all integers in a infinite series:

$$0, -1, 1, -2, 2, \ldots, -n, n, \ldots$$

the set of all integers (denoted usually by "$\mathbb{Z}$") is of cardinality $\aleph_0$, i.e., is countable. □

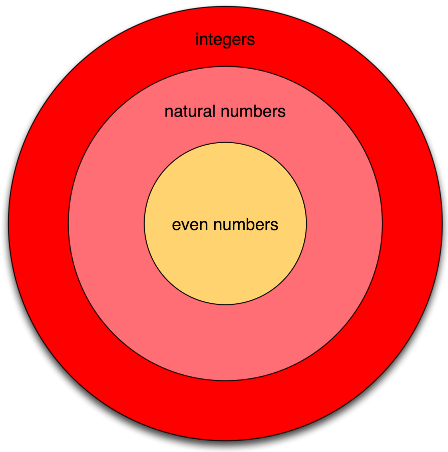

Figure 3.1: Wonders of the infinite - to be continued ...

These facts should amaze us since they imply that all of the aforementioned sets including:

- even numbers
- natural numbers
- integers

are of the same size, i.e., they have the same number of elements, despite the fact that each of them is properly contained in the next one - see figure 3.1.

The limit of our amazement is given by fact 4:

Fact 4. *The set of all rational numbers is countable.*

Proof. First let us build the following table - table 3.1.

$\frac{0}{1}$	$\frac{0}{2}$	...
$\frac{1}{1}$	$\frac{1}{2}$	...
$\frac{2}{1}$	$\frac{2}{2}$	...
...	...	...

Table 3.1: How to arrange all rational numbers in a series?

Note that the table collects all positive rational numbers; in fact, it contains also some repetitions.

Now it is crucial to see that one can arrange all cells from this table in a series - just "read them off" from the diagonals:

That is to say:

first	second	fourth	...
third	fifth	...	
sixth	ninth	...	
...	...	...	

Table 3.2: How to arrange all rational numbers in a series? (2)

$$\begin{aligned} \text{first} &\text{ is } \tfrac{0}{1} \\ \text{second} &\text{ is } \tfrac{0}{2} \\ \text{third} &\text{ is } \tfrac{1}{1} \\ \ldots &\text{ etc.} \end{aligned}$$

If you find a duplicated number in this series, just throw it away. Consequently, the series will contain all positive rational numbers.

In order to arrange all rational numbers you simply add 0 at the beginning of the series and handle all negative rationals in a similar way you handle integers.

In sum, one can arrange all rational number in a series and this means that the set of all rational numbers (denoted usually by "$\mathbb{Q}$") is countable. □

The above facts and their proofs may lead to the conclusion that all infinite sets are countable. Fact 5 shows that this conclusion is false.

Fact 5. *The set of all real numbers, i.e., $\Re$, is* not *countable.*

Proof. To show that $\Re$ is uncountable, we first write real numbers in their decimal (infinite) representations, e.g.,

- 1 as $1.00\ldots$
- $1\frac{1}{4}$ as $1.2500\ldots$
- $\frac{1}{3}$ as $0.33(3)$
- π as $3.14159265\ldots$
- etc.

Each real number will be represented as

$$i.f_1 f_2 \ldots,$$

where

1. i is an integer
2. each f_k is a digit

For example, for $1\frac{1}{4}$ we have

- $i = 0$
- $f_1 = 3$
- $f_2 = 3$
- etc.

Assume to the contrary that $\Re$ is countable. This implies that one can arrange all real numbers in an infinite series:

1. r_1
2. r_2
3. $\ldots$
4. r_j
5. $\ldots$

More precisely speaking, we will arrange the decimal representations of the reals in this series:

1. $r_1 = i^1.f_1^1 f_2^1 \ldots$
2. $r_2 = i^2.f_1^2 f_2^2 \ldots$

$\ldots$ $\ldots$

j $r_j = i^j.f_1^j f_2^j \ldots$

$\ldots$ $\ldots$

By its definition this series contains *all* real numbers.
Now consider the following number

$$r_0 = 0.f_1^0 f_2^0 \ldots,$$

where

$$f_j^0 = \begin{cases} 0 & \text{if } f_j^j \neq 0 \\ 1 & \text{if } f_j^j = 0 \end{cases} \tag{3.6}$$

Note that this definition is sufficiently perverse so that r_0 is not in our series.

1. r_0 cannot occupy position 1 in the series because it differs from r_1 in the first decimal place
2. r_0 cannot occupy position 2 in the series because it differs from r_2 in the second decimal place

$\ldots$ $\ldots$

j r_0 cannot occupy position j in the series because it differs from r_j in the jth decimal place

... ...

So this series does not contain, after all, all real numbers.

□

3.4.3 Infinity comparison

As before it is relatively easy to explain what it means that one (finite) number is greater or smaller than another. The things get more complicated when we are up to compare infinite numbers.

$$\mathfrak{n} \leq \mathfrak{m} \triangleq \exists X, Y[|X| = \mathfrak{n} \wedge |Y| = \mathfrak{m} \wedge \exists Z \subseteq Y X \sim Z]. \tag{3.7}$$

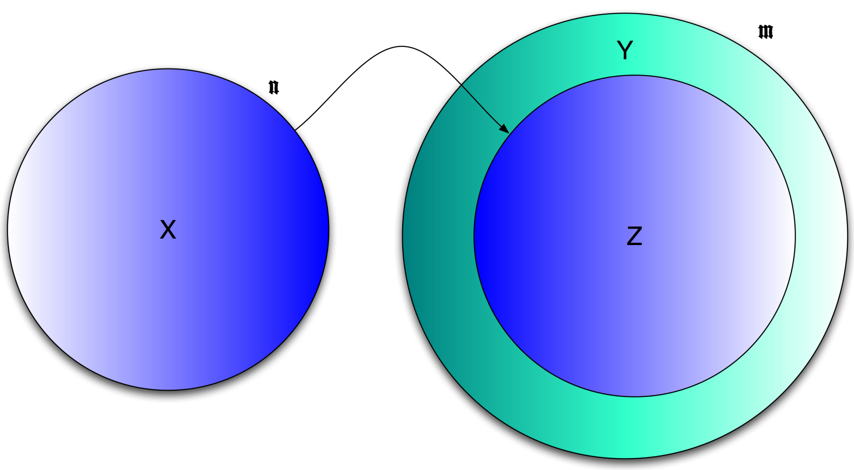

Figure 3.2: Set comparison

$$\mathfrak{n} < \mathfrak{m} \triangleq \mathfrak{n} \leq \mathfrak{m} \wedge \mathfrak{n} \neq \mathfrak{m}. \tag{3.8}$$

One of the most important theorems in set theory is Cantor-Bernstein theorem:

Theorem 2.

$$\mathfrak{n} \leq \mathfrak{m} \wedge \mathfrak{m} \leq \mathfrak{n} \rightarrow \mathfrak{n} = \mathfrak{m}.$$

Later one we will be able to compare quite a lot of cardinal numbers, but one we can make the following observation:

$$\aleph_0 < \mathfrak{c}. \tag{3.9}$$

Proof. Given fact 5 the proof is obvious. □

3.4.4 How to calculate with cardinal numbers

Addition

$$\mathfrak{k} = \mathfrak{n} + \mathfrak{m} \equiv \exists X, Y \ [|X| = \mathfrak{n} \wedge |Y| = \mathfrak{m} \wedge X \cap Y = \emptyset \wedge |X \cup Y| = \mathfrak{k}]. \tag{3.10}$$

General properties of the operation of addition:

$$\mathfrak{n} + \mathfrak{m} = \mathfrak{m} + \mathfrak{n}. \tag{3.11}$$

$$\mathfrak{n} + (\mathfrak{m} + \mathfrak{k}) = (\mathfrak{n} + \mathfrak{m}) + \mathfrak{k}. \tag{3.12}$$

$$\mathfrak{n} \leq \mathfrak{n} + \mathfrak{m}. \tag{3.13}$$

Particular properties of the operation of addition:

$$\text{If } \mathfrak{n} \text{ is a natural number, then } \mathfrak{n} + \aleph_0 = \aleph_0. \tag{3.14}$$

$$\aleph_0 + \aleph_0 = \aleph_0. \tag{3.15}$$

$$\mathfrak{c} + \aleph_0 = \mathfrak{c}. \tag{3.16}$$

$$\mathfrak{c} + \mathfrak{c} = \mathfrak{c}. \tag{3.17}$$

Multiplication First we should refresh our memory of the definition of Cartesian product - see definition 1.50 on page 24.

$$\mathfrak{k} = \mathfrak{n} * \mathfrak{m} \equiv \exists X, Y \ [|X| = \mathfrak{n} \wedge |Y| = \mathfrak{m} \wedge |X \times Y| = \mathfrak{k}]. \tag{3.18}$$

General properties of the operation of multiplication:

$$\mathfrak{n} * \mathfrak{m} = \mathfrak{m} * \mathfrak{n}. \tag{3.19}$$

$$\mathfrak{n} * (\mathfrak{m} * \mathfrak{k}) = (\mathfrak{n} * \mathfrak{m}) * \mathfrak{k}. \tag{3.20}$$

General properties of the operation of addition and multiplication:

$$\mathfrak{n} * (\mathfrak{m} + \mathfrak{k}) = \mathfrak{n} * \mathfrak{m} + \mathfrak{n} * \mathfrak{k}. \tag{3.21}$$

Particular properties of the operation of addition:

$$\text{If } \mathfrak{n} \text{ is a natural number, then } \mathfrak{n} * \aleph_0 = \aleph_0. \tag{3.22}$$

$$\aleph_0 * \aleph_0 = \aleph_0. \tag{3.23}$$

$$\mathfrak{c} * \aleph_0 = \mathfrak{c}. \tag{3.24}$$

$$\mathfrak{c} * \mathfrak{c} = \mathfrak{c}. \tag{3.25}$$

Exponentiation As one may expect, the definition of exponentiation is more complex than the previous two definitions.

First, we need one auxiliary notion. If X and Y are two sets, then "X^Y" denotes the set of all functions from Y into X.

$$\mathfrak{k} = \mathfrak{n}^{\mathfrak{m}} \equiv \exists X, Y\ [|X| = \mathfrak{n} \wedge |Y| = \mathfrak{m} \wedge |X^Y| = \mathfrak{k}]. \tag{3.26}$$

General properties of the operation of exponentiation:

$$\mathfrak{n}^0 = 1. \tag{3.27}$$
$$1^{\mathfrak{n}} = 1. \tag{3.28}$$
$$\mathfrak{n}^1 = \mathfrak{n}. \tag{3.29}$$
$$\tag{3.30}$$

Combined general properties of exponentiation, multiplication, and addition:

$$\mathfrak{n}^{\mathfrak{k}} * \mathfrak{n}^{\mathfrak{m}} = \mathfrak{n}^{\mathfrak{k}+\mathfrak{m}}. \tag{3.31}$$
$$(\mathfrak{n} * \mathfrak{m})^{\mathfrak{k}} = \mathfrak{n}^k * \mathfrak{m}^{\mathfrak{k}}. \tag{3.32}$$
$$(\mathfrak{n}^{\mathfrak{m}})^{\mathfrak{k}} = \mathfrak{n}^{\mathfrak{m}*\mathfrak{k}}. \tag{3.33}$$

Particular properties of exponentiation:

$$2^{\aleph_0} = \mathfrak{c}. \tag{3.34}$$

Proof. To prove 3.34 first we represent all real numbers by means of their binary expansions. Each such expansion is an infinite series of 0s and 1s, for example:

- $0_{10} = 0.00\ldots_2$
- $10_{10} = 1010.00\ldots_2$
- $\frac{1}{4}_{10} = 0.0100\ldots_2$

Each such series can be seen as a map (function) from $\mathbb{N}$ into $\{0, 1\}$:

- for $0.00\ldots_2$:
  1. $f(0) = 0$
  2. $f(1) = 0$
  3. ...
- for $10 = 1010.00\ldots_2$
  1. $f(0) = 1$
  2. $f(1) = 0$
  3. $f(2) = 1$
  4. $f(3) = 0$

5. $f(4) = 0$
6. ...

- for $0.0100\ldots_2$

1. $f(0) = 0$
2. $f(1) = 0$
3. $f(1) = 1$
4. $f(3) = 0$
5. $f(4) = 0$
6. ...

Thus the number of real numbers is equal to the numbers of functions from $\mathbb{N}$ into $\{0, 1\}$. The former is $\mathfrak{c}$. The latter is $2^{\aleph_0}$. Consequently, $2^{\aleph_0} = \mathfrak{c}$. $\square$

3.4.5 Cantor's theorem

You can wonder how many types of infinity there are. You have already had an opportunity to get acquainted with two: $\aleph_0$ and $\mathfrak{c}$. Are there any other infinities? The answer to this question is provided by Cantor's theorem.

$$|X| < |2^X| \qquad \text{(Cantor's theorem)}$$

Proof. In order to show that Cantor's theorem is true we need to prove two cases:

1. $|X| \leq |\wp(X)|$
2. $|X| \neq |\wp(X)|$

We will start with the first case because it is easier. To show that $|X| \leq |2^X|$ we need to find a subset Y of $\wp(X)$ that is equinumerous to X (see definition 3.7 on page 45). But this is easy - just pick up the set of all singletons from X:

$$Y = \{\{x\} : x \in X\}.$$

In order to show that $|X| \neq |\wp(X)|$ assume otherwise, i.e., assume that

$$|X| = |\wp(X)|.$$

This implies (see definitions on page 40) that there exists a bijective mapping f from X onto $\wp(X)$. Consider now set Z:

$$Z = \{x \in X : x \notin f(x)\}.$$

Is Z a value of f? On the one hand, it should be, i.e.,

$$\exists x\ Z = f(x),$$

because f is surjective. On the other hand, it cannot be. Assume otherwise, i.e., let

$$f(x_0) = Z,$$

for some $x_0 \in X$. One can now derive the following inconsistency:

$$x_0 \in Z \wedge x_0 \notin Z.$$

In order to show this we consider two cases.

1. $x_0 \in Z$
   Then, by definition of Z, $x_0 \notin f(x_0)$, i.e., $x_0 \notin Z$.

2. $x_0 \notin Z$
   Then, by definition of Z, $x_0 \notin f(x_0)$, i.e., $x_0 \in Z = f(x_0)$.

□

The answer that Cantor's theorem gives to the question "How many infinities are there?" is thus "There is an infinite number of infinities". Suppose that $X = \mathbb{N}$. Then Cantor's theorem implies that each item the following series is smaller than any subsequent item:

1. $\mathbb{N}$
2. $\wp(\mathbb{N})$
3. $\wp(\wp(\mathbb{N}))$

... ...

In other words Cantor's theorem implies that

$$\aleph_0 < 2^{\aleph_0} < 2^{2^{\aleph_0}} < 2^{2^{2^{\aleph_0}}} < \ldots \tag{3.35}$$

This fact, i.e., that there is an infinite number of infinite numbers, was dubbed by Cantor as "set-theoretical paradise".