Analytic geometry

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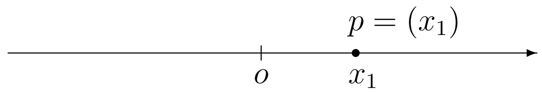
Preliminaries

The teaching script is a record of lectures of the course Linear algebra, which author have on KUL for informatics students. The course covers 15 hours, which means that there is not time for many interesting topics. First there is Cartesian space $\mathbb{R}^n$ and vectors in it discussed, and next there are presented subsets of $\mathbb{R}^2$ and $\mathbb{R}^3$ such as lines, planes and conics. Discussed notions are given in understanding form and often illustrated by examples. Author hopes that teaching script will by helpfull for student.

1. Cartesian space $\mathbb{R}^n$

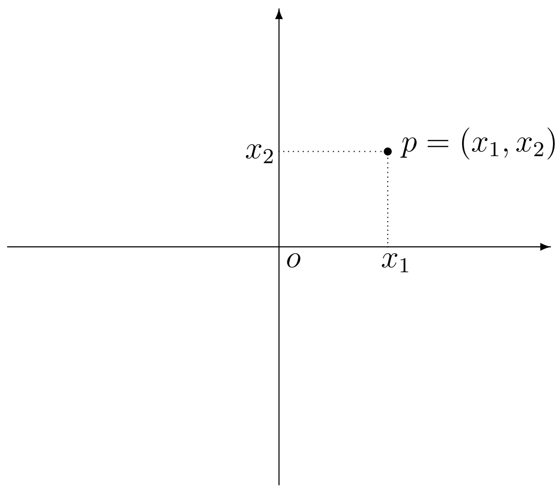
Cartesian coordinates on the line:



On the line we choose an arbitrary point o as the origin. It divides the line into two halflines. Regarding one of them as the positive halfline and the other as negative halfline, we obtain the axis. To any point p we assign a number x_1 called its Cartesian coordinate. In that way we get the **Cartesian space** $\mathbb{R}^1$. In that space we have the following formula of the distance of two points $x, y \in \mathbb{R}^1$:

$$\rho(x, y) = |x - y| .$$

Cartesian coordinates on the plane:

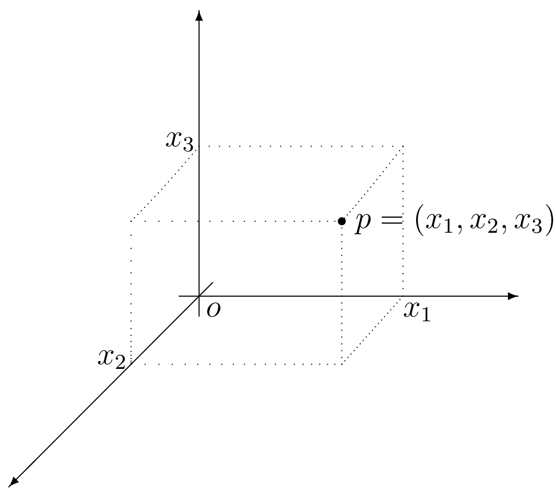


On the plane let us consider two lines intersecting at a point o as the origin and on each of them let us fix Cartesian coordinates. We obtain the axes, which form the Cartesian system of coordinates. We write $p = (x_1, x_2)$ and numbers x_1, x_2 we call Cartesian coordinates of the point p.

If axes are perpendicular, then the Cartesian coordinates are called rectangular. In that way we get the **Cartesian space** $\mathbb{R}^2$. In that space we have the following formula of the distance of two points $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$:

$$\rho(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

Cartesian coordinates in the space:



In the space let us take three lines not lying in one plane and passing through one point o as the origin, and on each of them let us fix Cartesian coordinates. We obtain the axes, which form the Cartesian system of coordinates. We write $p = (x_1, x_2, x_3)$ and numbers x_1, x_2, x_3 we call Cartesian coordinates of the point p.

If each axis is perpendicular to both the remaining ones, then the system is called rectangular. In that way we get the **Cartesian space** $\mathbb{R}^3$. In that space we have the following formula of the distance of two points $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathbb{R}^3$:

$$\rho(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}.$$

Definition. (**Metric space**) Let X be a set and $\rho : X \times X \to [0, \infty)$ be a function. A *metric space* is a pair (X, ρ) such that

1) $\bigwedge_{x,y \in X} \rho(x, y) = \rho(y, x)$,
2) $\bigwedge_{x,y \in X} \rho(x, y) = 0 \iff x = y$,
3) $\bigwedge_{x,y,z \in X} \rho(x, y) + \rho(y, z) \geq \rho(x, z)$.

Elements of X are called points, ρ is called a metrics and $\rho(x, y)$ is a distance of points x, y.

Definition. (**n-dimensional Cartesian space**) An n-dimensional Cartesian space is the set

$$\mathbb{R}^n = \{(x_1, \ldots, x_n) : x_i \in \mathbb{R}\}$$

together with a metrics $\rho : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ given by formula

$$\rho((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

Thus $(\mathbb{R}^n, \rho)$ is a metric space.

Exercise. Show that a function ρ defined above is a metrics.

Definition. Let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Define

$$x + y \underset{df}{=} (x_1 + y_1, \ldots, x_n + y_n) \quad - \quad \text{an addition of points } x, y,$$

$$-x \underset{df}{=} (-x_1, \ldots, -x_n)$$

$$x - y \underset{df}{=} x + (-y) \quad - \quad \text{a subtraction of points } x, y,$$

$$tx \underset{df}{=} (tx_1, \ldots, tx_n) \quad - \quad \text{a multiplication of a point } x \text{ by a number } t,$$

$$x \cdot y \underset{df}{=} \sum_{i=1}^{n} x_i y_i \quad - \quad \text{a scalar multiplication of points } x, y,$$

$$x^1 = x, x^{k+1} \underset{df}{=} x^k \cdot x \quad - \quad \text{a power of a point } x,$$

$$0 \underset{df}{=} (0, \ldots, 0).$$

Theorem. Let $x, y, z \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Then

1) $x + y = y + x$,
2) $(x + y) + z = x + (y + z)$,
3) $t(x + y) = tx + ty$,
4) $tx = 0 \Leftrightarrow t = 0 \vee x = 0$,
5) $x \cdot y = y \cdot x$,
6) $\sim (x \cdot y) \cdot z = x \cdot (y \cdot z)$,
7) $(tx) \cdot y = t(x \cdot y)$,
8) $x \cdot (y + z) = x \cdot y + x \cdot z$,
9) $(tx)^k = t^k x^k$,
10) $\sim (x \cdot y)^k = x^k \cdot y^k$,
11) $(x \cdot y)^2 \leq x^2 \cdot y^2$ – Cauchy-Schwarz inequality.

Proof. Easy. □

Definition. Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. A *modulus* of a point x is a number:

$$|x| \underset{df}{=} \rho(x, 0) = \sqrt{\sum_{i=1}^{n} x_i^2}$$

(it is the distance of a point x and point 0).

Theorem. Let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Then

1) $x^2 = |x|^2 = \sum_{i=1}^n x_i^2$,
2) $\rho(x, y) = |x - y| = \sqrt{(x-y)^2}$,
3) $|x| \geq 0$,
4) $|x| = |-x|$,
5) $|x| = 0 \Leftrightarrow x = 0$,
6) $|tx| = |t|\,|x|$,
7) $|x \cdot y| \leq |x| \cdot |y|$,
8) $|x + y| \leq |x| + |y|$,
9) $|x| - |y| \leq |x - y|$,
10) $(x + y)^2 = x^2 + 2x \cdot y + y^2$,
11) $(x - y)^2 = x^2 - 2x \cdot y + y^2$,
12) $x^2 - y^2 = (x - y) \cdot (x + y)$.

Proof. 1) – 5) Easy.

6) $|tx| = \sqrt{\sum_{i=1}^n (tx_i)^2} = \sqrt{t^2 \sum_{i=1}^n x_i^2} = |t| \sqrt{\sum_{i=1}^n x_i^2} = |t|\,|x|$.

7) $|x \cdot y| = \sqrt{\sum_{i=1}^n (x_i y_i)^2} = \sqrt{\sum_{i=1}^n x_i^2 y_i^2} \leq \sqrt{\sum_{i=1}^n x_i^2 \cdot \sum_{i=1}^n y_i^2} = \sqrt{\sum_{i=1}^n x_i^2} \cdot \sqrt{\sum_{i=1}^n y_i^2} = |x| \cdot |y|$ (by Cauchy-Schwarz inequality).

8) $|x + y| = |x - (-y)| = \rho(x, -y) \leq \rho(x, 0) + \rho(0, -y) = \rho(x, 0) + \rho(0, y) = |x| + |y|$.

9) $|x| = |y + (x - y)| \leq |y| + |x - y|$, whence $|x| - |y| \leq |x - y|$.

10), 11) and 12) follow from 8) of previous theorem. □

Definition. Let (X, ρ) be a metric space and let $a, b \in X$. A *metric segment* is a set:

$$\langle a, b \rangle \underset{df}{=} \{x \in X : \rho(a, x) + \rho(x, b) = \rho(a, b)\}.$$

Definition. Let (X, ρ) be a metric space and let $a, b, c \in X$. Then

$$c \text{ is a } \textit{centre} \text{ of a segment } \langle a, b \rangle \ \underset{df}{\Leftrightarrow} \ \rho(a, c) = \rho(b, c) = \frac{1}{2}\rho(a, b).$$

Theorem. Let $a, b \in \mathbb{R}^n$. Then there exists exactly one centre of a segment $\langle a, b \rangle$; it is a point $c = \frac{1}{2}(a + b)$.

Proof. If $a = b$, then Theorem is obvious. Let $a \neq b$. We have

$$\rho(a, c) = |a - c| = \left| a - \frac{1}{2}(a + b) \right| = \frac{1}{2}|a - b| = \frac{1}{2}|b - a| = \left| b - \frac{1}{2}(a + b) \right| = |b - c| = \rho(b, c).$$

Hence c is a centre of a segment $\langle a, b \rangle$.

Let $d = c + x$ be also a centre of a segment $\langle a, b \rangle$. Then

$$\rho(a, d) = \frac{1}{2}\rho(a, b) = \frac{1}{2}|a - b| = |a - d| = \left| a - \frac{1}{2}a - \frac{1}{2}b - x \right| = \left| \frac{1}{2}a - \frac{1}{2}b - \frac{1}{2} \cdot 2x \right| = \frac{1}{2}|a - b - 2x|,$$

that is, $|a - b| = |a - b - 2x|$.

Similarly,

$$\rho(b,d) = \frac{1}{2}\left|a-b\right| = |d-b| = \frac{1}{2}\left|a-b+2x\right|,$$

whence $|a-b| = |a-b+2x|$.

Thus,

$$\left|a-b-2x\right|^2 = \left|a-b+2x\right|^2,$$

that is,

$$(a-b)^2 - 4x(a-b) + 4x^2 = (a-b)^2 + 4x(a-b) + 4x^2,$$

whence

$$x(a-b) = 0.$$

Now, $a-b \neq 0$ (since $a \neq b$), so $x = 0$.

Thus, $d = c$. $\square$

Definition. Let $A \subseteq \mathbb{R}^n$. Then

$$A \text{ is } \textit{convex} \underset{df}{\Leftrightarrow} \bigwedge_{a,b\in A} \langle a,b\rangle \subseteq A.$$

Conclusion. A segment in $\mathbb{R}^n$ is a convex set.

2. Vectors in space $\mathbb{R}^n$

Definition. A *localized vector* in $\mathbb{R}^n \underset{df}{=}$ an ordered pair of points in $\mathbb{R}^n$.

Denotation: $\overrightarrow{ab}$ for $a, b \in \mathbb{R}^n$, a – the initial point of $\overrightarrow{ab}$, b – the end-point of $\overrightarrow{ab}$.

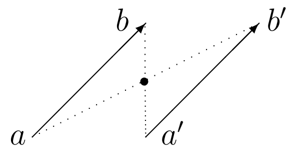
Definition. Coordinates of a localized vector $\overrightarrow{ab} \underset{df}{=}$ coordinates of a point $b - a$.

If $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in \mathbb{R}^n$, then $\overrightarrow{ab} = [b_1 - a_1, \ldots, b_n - a_n]$.

Definition. Let $a, b, a', b' \in \mathbb{R}^n$. Then

$$\overrightarrow{ab} = \overrightarrow{a'b'} \underset{df}{\Leftrightarrow} \overrightarrow{ab} \text{ and } \overrightarrow{a'b'} \text{ have the same coordinates } \underset{df}{\Leftrightarrow} b - a = b' - a'$$

$$\Leftrightarrow a' + b = a + b' \Leftrightarrow \frac{1}{2}(a' + b) = \frac{1}{2}(a + b')$$



(two localized vectors $\overrightarrow{ab}$ and $\overrightarrow{a'b'}$ are equal iff the centres of $\langle a', b\rangle$ and $\langle a, b'\rangle$ coinicide).

Theorem. The relation of equality of localized vectors is an equivalence relation.

Proof. Easy. □

Definition. A *free vector* (*vector*) in $\mathbb{R}^n \underset{df}{=}$ an equivalence class of the relation of equality of localized vectors,

that is,

$$\left[\overrightarrow{ab}\right] = \left\{\overrightarrow{cd} : \overrightarrow{ab} = \overrightarrow{cd}\right\} \quad - \quad \text{a free vector with a representative } \overrightarrow{ab}.$$

Denotation: $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \ldots$ (small gothic letters).

Remark. All representatives of a free vector have the same coordinates.

Definition. Coordinates of a free vector $\underset{df}{=}$ coordinates of its representative.

Definition. Let $\mathfrak{a}, a, b \in \mathbb{R}^n$ and $\overrightarrow{ab} \in \mathfrak{a}$. Then a *length* of a vector $\mathfrak{a}$ is a number:

$$|\mathfrak{a}| \underset{df}{=} \rho(a, b).$$

If $\mathfrak{a} = [\alpha_1, \ldots, \alpha_n]$, then $|\mathfrak{a}| = \sqrt{\sum_{i=1}^{n} \alpha_i^2}$.

Definition. A *versor* $\underset{df}{=}$ a vector of length 1.

Theorem. (**On localization of a free vector at a point**) Every free vector in $\mathbb{R}^n$ can be uniquely localized at an arbitrary point $a \in \mathbb{R}^n$.

Proof. Take $\mathfrak{a}, a \in \mathbb{R}^n$. We search a point $b \in \mathbb{R}^n$ such that $\overrightarrow{ab} \in \mathfrak{a}$. Let $\overrightarrow{cd} \in \mathfrak{a}$. Then

$$\overrightarrow{ab} = \overrightarrow{cd} \Leftrightarrow b - a = d - c \Leftrightarrow b = d - c + a. \;\square$$

Theorem. For every free vector $\mathfrak{a} \in \mathbb{R}^n$ and every point $b \in \mathbb{R}^n$ there exists a unique representative of $\mathfrak{a}$ with the end-point b.

Proof. Similar (we calculate a). $\square$

Definition. Let $\mathfrak{a} = [\alpha_1, \dots, \alpha_n], \mathfrak{b} = [\beta_1, \dots, \beta_n] \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Define

$$\mathfrak{a} + \mathfrak{b} \underset{df}{=} [\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n] \quad - \quad \text{an addition of vectors } \mathfrak{a}, \mathfrak{b},$$

$$-\mathfrak{a} \underset{df}{=} [-\alpha_1, \dots, -\alpha_n] \quad - \quad \text{an opposite vector for } \mathfrak{a},$$

$$\mathfrak{a} - \mathfrak{b} \underset{df}{=} [\alpha_1 - \beta_1, \dots, \alpha_n - \beta_n] \quad - \quad \text{a subtraction of vectors } \mathfrak{a}, \mathfrak{b},$$

$$t\mathfrak{a} \underset{df}{=} [t\alpha_1, \dots, t\alpha_n] \quad - \quad \text{a multiplication of a vector } \mathfrak{a} \text{ by a number } t,$$

$$\mathfrak{a} \cdot \mathfrak{b} \underset{df}{=} \sum_{i=1}^{n} \alpha_i \beta_i \quad - \quad \text{a scalar product of vectors } \mathfrak{a}, \mathfrak{b}.$$

Remark. We will write $\mathfrak{a} \cdot \mathfrak{a} = \mathfrak{a}^2$.

Theorem. Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Then

1) $\mathfrak{a} \cdot \mathfrak{b} = \mathfrak{b} \cdot \mathfrak{a}$,
2) $(t\mathfrak{a}) \cdot \mathfrak{b} = t(\mathfrak{a} \cdot \mathfrak{b})$,
3) $\mathfrak{a}^2 = |\mathfrak{a}|^2$,
4) $\mathfrak{a} \cdot (\mathfrak{b} + \mathfrak{c}) = \mathfrak{a} \cdot \mathfrak{b} + \mathfrak{a} \cdot \mathfrak{c}$,
5) $-|\mathfrak{a}|\,|\mathfrak{b}| \leq \mathfrak{a} \cdot \mathfrak{b} \leq |\mathfrak{a}|\,|\mathfrak{b}|$.

Proof. Easy. Point 5) follows from Cauchy-Schwartz inequality. $\square$

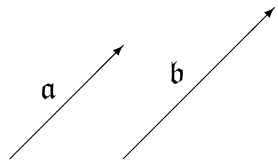
Theorem. Let $a, b, c, \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathbb{R}^n$. Then

$$\overrightarrow{ab} \in \mathfrak{a} \;\wedge\; \overrightarrow{bc} \in \mathfrak{b} \;\Rightarrow\; \overrightarrow{ac} \in [\mathfrak{a} + \mathfrak{b}].$$

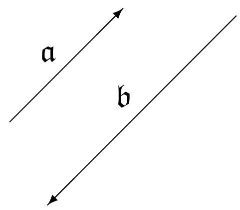
Proof. Easy. $\square$

Definition. Let $\mathfrak{a}, \mathfrak{b} \in \mathbb{R}^n$. Then

$$\mathfrak{a}, \mathfrak{b} \text{ are } \textit{equally parallel}, \; \mathfrak{a} \upharpoonleft\upharpoonright \mathfrak{b} \underset{df}{\Leftrightarrow} |\mathfrak{a}| + |\mathfrak{b}| = |\mathfrak{a} + \mathfrak{b}|$$



$$\mathfrak{a}, \mathfrak{b} \text{ are } \textit{oppositely parallel}, \ \ \mathfrak{a} \upharpoonleft\downharpoonright \mathfrak{b} \underset{df}{\Leftrightarrow} |\mathfrak{a}| + |\mathfrak{b}| = |\mathfrak{a} - \mathfrak{b}|$$



$$\mathfrak{a}, \mathfrak{b} \text{ are } \textit{parallel}, \ \ \mathfrak{a} \parallel \mathfrak{b} \underset{df}{\Leftrightarrow} \mathfrak{a} \upharpoonleft\upharpoonright \mathfrak{b} \ \vee \ \mathfrak{a} \upharpoonleft\downharpoonright \mathfrak{b}$$

Theorem. Let $\mathfrak{a}, \mathfrak{b} \in \mathbb{R}^n$ be such that $\mathfrak{a} \neq 0 \neq \mathfrak{b}$. Then

$$\mathfrak{a} \parallel \mathfrak{b} \Leftrightarrow \bigvee_{t \neq 0} \mathfrak{b} = t\mathfrak{a}$$

$$\text{and} \quad t > 0 \Rightarrow \mathfrak{a} \upharpoonleft\upharpoonright \mathfrak{b},$$

$$t < 0 \Rightarrow \mathfrak{a} \upharpoonleft\downharpoonright \mathfrak{b}.$$

Proof. Easy. □

Theorem. In the set of nonzero vectors in $\mathbb{R}^n$ relations $\parallel$ and $\upharpoonleft\upharpoonright$ are equivalence relations.

Proof. Easy. □

Definition. Let $\mathfrak{a} \in \mathbb{R}^n$.

A *direction* of a vector $\mathfrak{a} \underset{df}{=}$ an equivalence class of the relation $\parallel$ with a representative $\mathfrak{a}$, that is,

$$\mathcal{K}(\mathfrak{a}) = \{\mathfrak{b} : \mathfrak{b} \parallel \mathfrak{a} \ \wedge \ \mathfrak{b} \neq 0\}.$$

A *sense* of a vector $\mathfrak{a} \underset{df}{=}$ an equivalence class of the relation $\upharpoonleft\upharpoonright$ with a representative $\mathfrak{a}$, that is,

$$\mathcal{Z}(\mathfrak{a}) = \{\mathfrak{b} : \mathfrak{b} \upharpoonleft\upharpoonright \mathfrak{a} \ \wedge \ \mathfrak{b} \neq 0\}.$$

We have: $\mathcal{Z}(\mathfrak{a}) \subseteq \mathcal{K}(\mathfrak{a})$.

Remark. Let $\mathfrak{a}, \mathfrak{b} \in \mathbb{R}^n$ be such that $\mathfrak{a} \neq 0 \neq \mathfrak{b}$. Since $-|\mathfrak{a}|\,|\mathfrak{b}| \leq \mathfrak{a} \cdot \mathfrak{b} \leq |\mathfrak{a}|\,|\mathfrak{b}|$, it follows that there is a unique number θ such that

$$\mathfrak{a} \cdot \mathfrak{b} = |\mathfrak{a}|\,|\mathfrak{b}| \cos\theta \quad \text{and} \quad 0 \leq \theta \leq \pi.$$

If $\mathfrak{a} = 0$ or $\mathfrak{b} = 0$, then θ is arbitrary such that $0 \le \theta \le \pi$.

Definition. Let $\mathfrak{a}, \mathfrak{b} \in \mathbb{R}^n$. A number $\sphericalangle(\mathfrak{a}, \mathfrak{b}) \in [0, \pi]$ such that

$$\cos(\sphericalangle(\mathfrak{a}, \mathfrak{b})) = \frac{\mathfrak{a} \cdot \mathfrak{b}}{|\mathfrak{a}|\,|\mathfrak{b}|}$$

is called an *angle in* $\mathbb{R}^n$ *between vectors* $\mathfrak{a}, \mathfrak{b}$.

Theorem. Let $\mathfrak{a}, \mathfrak{b} \in \mathbb{R}^n$. Then

1) $\sphericalangle(\mathfrak{a}, \mathfrak{b}) = \sphericalangle(\mathfrak{b}, \mathfrak{a})$,
2) $t, s > 0 \;\Rightarrow\; \sphericalangle(\mathfrak{a}, \mathfrak{b}) = \sphericalangle(t\mathfrak{a}, s\mathfrak{b})$,
3) $\sphericalangle(\mathfrak{a}, \mathfrak{b}) + \sphericalangle(-\mathfrak{a}, \mathfrak{b}) = \pi$,
4) $\sphericalangle(\mathfrak{a}, \mathfrak{b}) = \sphericalangle(-\mathfrak{a}, -\mathfrak{b})$.

Proof. Easy. □

Definition. Let $\mathfrak{a}, \mathfrak{b} \in \mathbb{R}^n$. Then

$$\mathfrak{a}, \mathfrak{b} \text{ are } \textit{perpendicular},\; \mathfrak{a} \perp \mathfrak{b} \underset{df}{\Leftrightarrow} \sphericalangle(\mathfrak{a}, \mathfrak{b}) = \frac{\pi}{2} \;\vee\; \mathfrak{a} = 0 \;\vee\; \mathfrak{b} = 0.$$

Theorem. Let $\mathfrak{a}, \mathfrak{b} \in \mathbb{R}^n$. Then

$$\mathfrak{a} \perp \mathfrak{b} \;\Leftrightarrow\; \mathfrak{a} \cdot \mathfrak{b} = 0.$$

Proof. Follows immediately from the formula $\mathfrak{a} \cdot \mathfrak{b} = |\mathfrak{a}|\,|\mathfrak{b}| \cos(\sphericalangle(\mathfrak{a}, \mathfrak{b}))$. □

Definition. (**Vector product in** $\mathbb{R}^3$) Let $\mathfrak{a}, \mathfrak{b} \in \mathbb{R}^3$, $\mathfrak{a} = [\alpha_1, \alpha_2, \alpha_3]$ and $\mathfrak{b} = [\beta_1, \beta_2, \beta_3]$. A vector product of $\mathfrak{a}$ and $\mathfrak{b}$ is a vector

$$\mathfrak{a} \times \mathfrak{b} \underset{df}{=} \left[\begin{vmatrix} \alpha_2 & \alpha_3 \\ \beta_2 & \beta_3 \end{vmatrix}, - \begin{vmatrix} \alpha_1 & \alpha_3 \\ \beta_1 & \beta_3 \end{vmatrix}, \begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix} \right].$$

Remark. If we denote by i, j, k versors of coordinate axes in $\mathbb{R}^3$, that is, $i = [1, 0, 0]$, $j = [0, 1, 0]$ and $k = [0, 0, 1]$, then

$$\mathfrak{a} \times \mathfrak{b} = \begin{vmatrix} i & j & k \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{vmatrix}.$$

Example. Determine $\mathfrak{a} \times \mathfrak{b}$ if $\mathfrak{a} = [1, 1, -1]$ and $\mathfrak{b} = [2, -1, 3]$.

Solution.

$$\mathfrak{a} \times \mathfrak{b} = \begin{vmatrix} i & j & k \\ 1 & 1 & -1 \\ 2 & -1 & 3 \end{vmatrix} = \left[\begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix}, - \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} \right] = [2, -5, -3].$$

Theorem. Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathbb{R}^3$. Then

1) $\mathfrak{a} \times \mathfrak{a} = 0$,
2) $\mathfrak{a} \times \mathfrak{b} = -\mathfrak{b} \times \mathfrak{a}$,
3) $\mathfrak{a} \times (\mathfrak{b} + \mathfrak{c}) = \mathfrak{a} \times \mathfrak{b} + \mathfrak{a} \times \mathfrak{c}$ and $(\mathfrak{a} + \mathfrak{b}) \times \mathfrak{c} = \mathfrak{a} \times \mathfrak{c} + \mathfrak{b} \times \mathfrak{c}$,
4) $t \cdot (\mathfrak{a} \times \mathfrak{b}) = (t \cdot \mathfrak{a}) \times \mathfrak{b} = \mathfrak{a} \times (t \cdot \mathfrak{b})$, where $t \in \mathbb{R}$,
5) $(\mathfrak{a} \times \mathfrak{b}) \cdot \mathfrak{c} = \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix}$, where $\mathfrak{a} = [\alpha_1, \alpha_2, \alpha_3]$, $\mathfrak{b} = [\beta_1, \beta_2, \beta_3]$, $\mathfrak{c} = [\gamma_1, \gamma_2, \gamma_3]$,
6) $\mathfrak{a} \times \mathfrak{b} = 0 \Leftrightarrow \mathfrak{a} \parallel \mathfrak{b}$,
7) $\mathfrak{a} \times \mathfrak{b} \perp \mathfrak{a}$ and $\mathfrak{a} \times \mathfrak{b} \perp \mathfrak{b}$,
8) $|\mathfrak{a} \times \mathfrak{b}| = |\mathfrak{a}|\,|\mathfrak{b}| \sin \sphericalangle(\mathfrak{a}, \mathfrak{b})$.

Proof. Points 1) – 5) follow from above Remark and definition.

6) We have

$$\mathfrak{a} \parallel \mathfrak{b} \;\Leftrightarrow\; \bigvee_{t \neq 0} \mathfrak{b} = t\mathfrak{a} \;\Leftrightarrow\; \bigvee_{t \neq 0} (t\mathfrak{a}) \times \mathfrak{b} = \mathfrak{b} \times \mathfrak{b} = 0 \;\Leftrightarrow\; \bigvee_{t \neq 0} t(\mathfrak{a} \times \mathfrak{b}) = 0 \;\Leftrightarrow\; \mathfrak{a} \times \mathfrak{b} = 0.$$

7) Follows from 5).

8) We have for $\mathfrak{a} = [\alpha_1, \alpha_2, \alpha_3]$ and $\mathfrak{b} = [\beta_1, \beta_2, \beta_3]$:

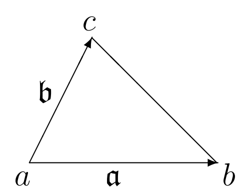
$$\begin{aligned} |\mathfrak{a} \times \mathfrak{b}|^2 &= (\alpha_2\beta_3 - \alpha_3\beta_2)^2 + (\alpha_1\beta_3 - \alpha_3\beta_1)^2 + (\alpha_1\beta_2 - \alpha_2\beta_1)^2 \\ &= (\alpha_1^2 + \alpha_2^2 + \alpha_3^2)(\beta_1^2 + \beta_2^2 + \beta_3^2) - (\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3)^2 \\ &= \mathfrak{a}^2\mathfrak{b}^2 - (\mathfrak{a} \cdot \mathfrak{b})^2 \\ &= (|\mathfrak{a}|\,|\mathfrak{b}|)^2 - (|\mathfrak{a}|\,|\mathfrak{b}|)^2 \cos^2 \sphericalangle(\mathfrak{a}, \mathfrak{b}) \\ &= (|\mathfrak{a}|\,|\mathfrak{b}| \sin \sphericalangle(\mathfrak{a}, \mathfrak{b}))^2, \end{aligned}$$

whence $|\mathfrak{a} \times \mathfrak{b}| = |\mathfrak{a}|\,|\mathfrak{b}| \sin \sphericalangle(\mathfrak{a}, \mathfrak{b})$. $\square$

Theorem. Let $a, b, c \in \mathbb{R}^3$, $\triangle(a, b, c)$ be a triangle with vertices a, b, c, $\mathfrak{a} = \left[\overrightarrow{ab}\right]$ and $\mathfrak{b} = \left[\overrightarrow{ac}\right]$. Then

$$|\triangle(a, b, c)| = \frac{1}{2}\,|\mathfrak{a} \times \mathfrak{b}|$$

(the area of a triangle).



Proof. We have the following Heron's formula

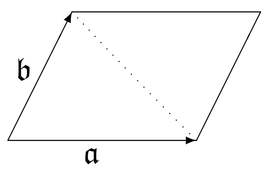
$$|\triangle(a, b, c)| = \frac{1}{4}\sqrt{s[s - 2\rho(b, c)][s - 2\rho(a, c)][s - 2\rho(a, b)]},$$

where $s = \rho(a,b) + \rho(a,c) + \rho(b,c)$.
Hence

$$
\begin{aligned}
|\triangle(a,b,c)| &= \frac{1}{4}\sqrt{(|\mathfrak{a}| + |\mathfrak{b}| + |\mathfrak{a}-\mathfrak{b}|)(|\mathfrak{a}| + |\mathfrak{b}| - |\mathfrak{a}-\mathfrak{b}|)(|\mathfrak{a}| - |\mathfrak{b}| + |\mathfrak{a}-\mathfrak{b}|)(-|\mathfrak{a}| + |\mathfrak{b}| + |\mathfrak{a}-\mathfrak{b}|)} \\
&= \frac{1}{2}\sqrt{(|\mathfrak{a}|\,|\mathfrak{b}| + \mathfrak{a}\cdot\mathfrak{b})(|\mathfrak{a}|\,|\mathfrak{b}| - \mathfrak{a}\cdot\mathfrak{b})} \\
&= \frac{1}{2}\sqrt{\mathfrak{a}^2\mathfrak{b}^2 - (\mathfrak{a}\cdot\mathfrak{b})^2} \\
&= \frac{1}{2}\,|\mathfrak{a}|\,|\mathfrak{b}|\sin\sphericalangle(\mathfrak{a},\mathfrak{b}).
\end{aligned}
$$

Thus $|\triangle(a,b,c)| = \frac{1}{2}\,|\mathfrak{a}\times\mathfrak{b}|$. $\square$

Conclusion. The number $|\mathfrak{a}\times\mathfrak{b}|$ is the area of a parallelogram built on vectors $\mathfrak{a}, \mathfrak{b} \in \mathbb{R}^3$:



3. Transformations of metric spaces

Definition. Let (X, ρ), $(Y, \overline{\rho})$ be metric spaces and $f : X \to Y$ be a function. Then

$$f \text{ is an } isometry \underset{df}{\Leftrightarrow} 1)\ f : X \xrightarrow{onto} Y,$$

$$2) \bigwedge_{x,x' \in X} \overline{\rho}(f(x), f(x')) = \rho(x, x').$$

Examples.

1. Translation: $a \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(x) = x + a$ for $x \in \mathbb{R}^n$. Then f is an isometry, since

$$\rho(f(x), f(x')) = \sqrt{(f(x) - f(x'))^2} = \sqrt{[(x + a) - (x' + a)]^2} = \sqrt{(x - x')^2} = \rho(x, x')$$

for $x, x' \in \mathbb{R}^n$.

2. Rotation of the plane $\mathbb{R}^2$: $\alpha \in \mathbb{R}$, $x = (x_1, x_2) \in \mathbb{R}^2$, $f : \mathbb{R}^2 \to \mathbb{R}^2$
$f(x) = (x_1 \cos\alpha - x_2 \sin\alpha, x_1 \sin\alpha + x_2 \cos\alpha)$ – rotation through the angle α
Then f is an isometry, since

$$\begin{aligned}\rho(f(x), f(x'))^2 &= [(x_1 - x_1') \cos\alpha - (x_2 - x_2') \sin\alpha]^2 + [(x_1 - x_1') \sin\alpha + (x_2 - x_2') \cos\alpha]^2 \\ &= (x_1 - x_1')^2 + (x_2 - x_2')^2 \\ &= \rho(x, x')^2\end{aligned}$$

for $x = (x_1, x_2), x' = (x_1', x_2') \in \mathbb{R}^2$.

Theorem. An isometry is a one-to-one transformation.

Proof. Let (X, ρ), $(Y, \overline{\rho})$ be metric spaces and $f : X \to Y$ be an isometry. Take $x, x' \in X$. Assume that $f(x) = f(x')$. Then

$$0 = \overline{\rho}(f(x), f(x')) = \rho(x, x') \;\Rightarrow\; x = x'. \;\square$$

Theorem. If $f : X \to Y$ is an isometry, then $f^{-1} : Y \to X$ is an isometry.

Proof. Let (X, ρ), $(Y, \overline{\rho})$ be metric spaces and $f : X \to Y$ be an isometry. Obviously, f^{-1} is onto (because f is onto). Let $y, y' \in Y$. There are $x, x' \in X$ such that $f^{-1}(y) = x$ and $f^{-1}(y') = x'$. Hence $y = f(x)$ and $y' = f(x')$. We have

$$\rho(f^{-1}(y), f^{-1}(y')) = \rho(x, x') = \overline{\rho}(f(x), f(x')) = \overline{\rho}(y, y'). \;\square$$

Theorem. Composition of two isometries is an isometry.

Proof. Let (X, ρ), $(Y, \overline{\rho})$, $(Z, \widehat{\rho})$ be metric spaces and $f : X \to Y$, $g : Y \to Z$ be isometries. So

$$\bigwedge_{x,x' \in X} \overline{\rho}(f(x), f(x')) = \rho(x, x')$$

and

$$\bigwedge_{y,y' \in Y} \widehat{\rho}(g(y), g(y')) = \overline{\rho}(y, y').$$

Then $gf : X \to Z$ and

$$\bigwedge_{x,x' \in X} \widehat{\rho}(gf(x), gf(x')) = \overline{\rho}(f(x), f(x')) = \rho(x, x'). \ \square$$

Definition. Let (X, ρ), $(Y, \overline{\rho})$ be metric spaces and $f : X \to Y$ be a function. Then

$$f \text{ is a } \textit{similarity} \underset{df}{\Leftrightarrow} 1)\ f : X \overset{onto}{\longrightarrow} Y,$$
$$2) \bigvee_{\lambda > 0} \bigwedge_{x,x' \in X} \overline{\rho}(f(x), f(x') = \lambda \rho(x, x').$$

Number λ is then called the ratio of similarity f.

Remark. Any isometry is a similarity with the ratio 1.

Example. Homothety with the ratio $c > 0$: $j_c : \mathbb{R}^n \to \mathbb{R}^n$, $j_c(x) = cx$ for $x \in \mathbb{R}^n$. Then j_c is a similarity with the ratio c, since

$$\rho(j_c(x), j_c(x')) = \sqrt{(j_c(x) - j_c(x'))^2} = \sqrt{(cx - cx')^2} = c\sqrt{(x - x')^2} = c\rho(x, x')$$

for $x, x' \in \mathbb{R}^n$.

Theorem. A similarity is a one-to-one transformation.

Proof. Let (X, ρ), $(Y, \overline{\rho})$ be metric spaces and $f : X \to Y$ be a similarity with the ratio $\lambda > 0$. Let $x, x' \in X$ and $f(x) = f(x')$. Then

$$0 = \overline{\rho}(f(x), f(x')) = \lambda \rho(x, x')$$

and

$$\lambda > 0 \ \Rightarrow \ \rho(x, x') = 0 \ \Rightarrow \ x = x'. \ \square$$

Theorem. If $f : X \to Y$ is a similarity with the ratio $\lambda > 0$, then $f^{-1} : Y \to X$ is a similarity with the ratio $\frac{1}{\lambda}$.

Proof. Let (X, ρ), $(Y, \overline{\rho})$ be metric spaces and $f : X \to Y$ be a similarity with the ratio $\lambda > 0$. Obviously, f^{-1} is onto (because f is onto). Let $y, y' \in Y$. There are $x, x' \in X$ such that $f^{-1}(y) = x$ and $f^{-1}(y') = x'$. Hence $y = f(x)$ and $y' = f(x')$. We have

$$\rho(f^{-1}(y), f^{-1}(y')) = \rho(x, x') = \frac{1}{\lambda}\overline{\rho}(f(x), f(x')) = \frac{1}{\lambda}\overline{\rho}(y, y').$$

Thus f^{-1} is a similarity with the ratio $\frac{1}{\lambda}$. $\square$

Theorem. Composition of two similarities is a similarity.

Proof. Let (X, ρ), $(Y, \overline{\rho})$, $(Z, \widehat{\rho})$ be metric spaces. Let $f : X \to Y$ be a similarity with the ratio λ_1, $g : Y \to Z$ be a similarity with the ratio λ_2. We will show that $gf : X \to Z$ is a similarity with the ratio $\lambda_1\lambda_2$. Let $x, x' \in X$ and $y, y' \in Y$. We know that

$$\overline{\rho}(f(x), f(x')) = \lambda_1 \rho(x, x')$$

and

$$\widehat{\rho}(g(y), g(y')) = \lambda_2 \overline{\rho}(y, y').$$

We have

$$\widehat{\rho}(gf(x), gf(x')) = \lambda_2 \overline{\rho}(f(x), f(x')) = \lambda_1\lambda_2 \rho(x, x'). \quad \square$$

Definition. Let (X, ρ), $(Y, \overline{\rho})$ be metric spaces. Then

X and Y are *isometric* $\underset{df}{\Leftrightarrow}$ there exists an isometry $f : X \to Y$.

X and Y are *similar* $\underset{df}{\Leftrightarrow}$ there exists a similarity $g : X \to Y$.

Remark. If X, Y are isometric, then they are similar. The converse is not true.

4. Lines, planes and hyperplanes in space $\mathbb{R}^n$

Definition. Let (X, ρ) be a metric space and $Y \subseteq X$. Then

$$(Y, \rho|Y \times Y) \underset{df}{=} \text{a subspace of a metric space } (X, \rho).$$

Definition.

A *line* $\underset{df}{=}$ a subspace of the space $\mathbb{R}^n$ isometric with $\mathbb{R}^1$.

Remark. Let $L \subseteq \mathbb{R}^n$. Then

L is a line $\Leftrightarrow$ L is isometric with $\mathbb{R}^1$ $\Leftrightarrow$ there exists an isometry $f : \mathbb{R}^1 \to L$ $\Leftrightarrow$ there exists an isometry $g : L \to \mathbb{R}^1$.

Remark. In $\mathbb{R}^1$ there exists a unique line. It is $\mathbb{R}^1$.

Theorem. (**On a line**) Through every two distinct points $a, b \in \mathbb{R}^n$ there passes exactly one line. It is the set $\{x(t) = (1-t)a + tb : t \in \mathbb{R}\} = L(a, b)$, where $x : \mathbb{R} \to \mathbb{R}^n$ is called the parametric presentation of a line $L(a, b)$.

Proof. Take $f : \mathbb{R}^1 \to L(a, b)$ such that $f(t) = x\left(\frac{t}{\rho(a,b)}\right)$, $t \in \mathbb{R}$. We have for $t, t' \in \mathbb{R}$:

$$\begin{aligned}\rho(f(t), f(t'))^2 &= \left[\left(1 - \frac{t}{\rho(a,b)}\right)a + \frac{t}{\rho(a,b)}b - \left(1 - \frac{t'}{\rho(a,b)}\right)a - \frac{t'}{\rho(a,b)}b\right]^2 \\ &= \left[\frac{(t-t')a - (t-t')b}{\rho(a,b)}\right]^2 = (t-t')^2 \\ &= \rho(t, t')^2.\end{aligned}$$

Hence f is an isometry, that is, $L(a, b)$ is a line. Moreover, $x(0) = a$ and $x(1) = b$ whence $a, b \in L(a, b)$.

Now we show that $L(a, b)$ is unique. Assume that there is a line K such that $a, b \in K$. We show that $K \subseteq L(a, b)$. Take an isometry $g : \mathbb{R}^1 \to K$. There are $\alpha, \beta \in \mathbb{R}$ such that $g(\alpha) = a$, $g(\beta) = b$ and $\alpha < \beta$. Take $c = g(\gamma) \in K$ such that $a \neq c \neq b$. Suppose that $\alpha < \beta < \gamma$. Then $|\beta - \alpha| + |\gamma - \beta| = |\gamma - \alpha|$. Hence $\rho(b, a) + \rho(c, b) = \rho(c, a)$, because g is an isometry. It follows

$$\left|\overrightarrow{ab}\right| + \left|\overrightarrow{bc}\right| = \left|\overrightarrow{ac}\right| = \left|\overrightarrow{ab} + \overrightarrow{bc}\right|,$$

so $\overrightarrow{ab} \parallel \overrightarrow{ac}$. Thus there exists $t \neq 0$ such that $c - a = t(b - a)$, whence $c = (1-t)a + tb = x(t) \in L(a, b)$.

Similarly when $\alpha < \gamma < \beta$ and $\gamma < \alpha < \beta$. Hence $K \subseteq L(a, b)$. Precisely, $K = L(a, b)$. $\square$

Remark. We will write the following parametric equation of $L(a,b)$:

$$L = L(a,b) : x(t) = (1-t)a + tb,\ t \in \mathbb{R}.$$

Definition. Let $\mathfrak{a}, a, b \in \mathbb{R}^n$ and let $L \subseteq \mathbb{R}^n$ be a line. Then

$\overrightarrow{ab}$ lies on $L \underset{df}{\Leftrightarrow} a, b \in L$.

$\mathfrak{a} \parallel L \underset{df}{\Leftrightarrow} \bigvee_{\overrightarrow{ab}} \overrightarrow{ab} \in \mathfrak{a} \wedge \overrightarrow{ab}$ lies on $L \Leftrightarrow \bigvee_{a,b \in L} \overrightarrow{ab} \in \mathfrak{a}$.

Definition. Let $\mathfrak{a} \in \mathbb{R}^n$ and let $L \subseteq \mathbb{R}^n$ be a line.

A *direction* of a line $L \underset{df}{=}$ a direction of a vector $\mathfrak{a} \parallel L$.

A *direction vector* of a line $L \underset{df}{=}$ a vector $\mathfrak{a} \parallel L$.

Theorem. (The second form of the parametric equation of a line in $\mathbb{R}^n$)

Let $\mathfrak{a}, a \in \mathbb{R}^n$ and let $L \subseteq \mathbb{R}^n$ be a line. Then

$$a \in L \wedge \mathfrak{a} \parallel L \wedge \mathfrak{a} \neq 0 \Rightarrow L : x(t) = a + t\mathfrak{a},\ t \in \mathbb{R}.$$

Proof. Let $a \in L$, $\mathfrak{a} \parallel L$ and $\mathfrak{a} \neq 0$. By Theorm on localization of a free vector at a point, a vector $\mathfrak{a}$ can be localized at a point a. Then there exists a point $b \in L$ (because $\mathfrak{a} \parallel L$) such that $\mathfrak{a} = \left[\overrightarrow{ab}\right]$. By Theorem on a line for $t \in \mathbb{R}$:

$$L : x(t) = (1-t)a + tb, \text{ so}$$
$$L : x(t) = a + t(b-a),$$
$$L : x(t) = a + t\left[\overrightarrow{ab}\right],$$
$$L : x(t) = a + t\mathfrak{a}. \quad \square$$

Remark. If $a = (a_1, \ldots, a_n) \in L$ and $\mathfrak{a} = [\alpha_1, \ldots, \alpha_n] \parallel L$, then a parametric equation of $L : x(t) = a + t\mathfrak{a},\ t \in \mathbb{R}$ has a form:

$$L : x(t) = (a_1 + t\alpha_1, \ldots, a_n + t\alpha_n),\ t \in \mathbb{R}.$$

For example, $L : x(t) = (1+2t, -1+3t)$, where $t \in \mathbb{R}$, is the line in $\mathbb{R}^2$ such that $a = (1,-1) \in L$ and $\mathfrak{a} = [2,3] \parallel L$, and $K : y(s) = (-1+s, 2-s, 3+2s)$, where $s \in \mathbb{R}$, is the line in $\mathbb{R}^3$ such that $a = (-1,2,3) \in K$ and $\mathfrak{a} = [1,-1,2] \parallel K$.

Definition. Let $L, K \subseteq \mathbb{R}^n$ be lines, $\mathfrak{a} \parallel L$ and $\mathfrak{b} \parallel K$. Then

$$L \parallel K \underset{df}{\Leftrightarrow} \mathfrak{a} \parallel \mathfrak{b} \Leftrightarrow \bigvee_{t \neq 0} \mathfrak{b} = t\mathfrak{a}.$$

$$L \perp K \underset{df}{\Leftrightarrow} \mathfrak{a} \perp \mathfrak{b} \Leftrightarrow \mathfrak{a} \cdot \mathfrak{b} = 0.$$

Definition. Let $\mathfrak{a} \in \mathbb{R}^2$ and let $L \subseteq \mathbb{R}^2$ be a line.

A *normal direction* of a line $L \underset{df}{=}$ a direction of a vector $\mathfrak{a} \perp L$.

A *normal vector* of a line $L \underset{df}{=}$ a vector $\mathfrak{a} \perp L$.

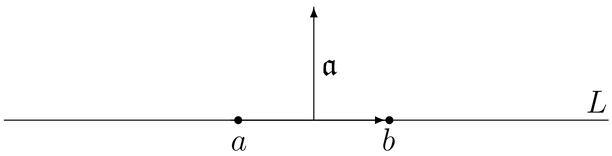
Theorem. For every point $a \in \mathbb{R}^2$ and every nonzero vector $\mathfrak{a} = [\alpha_1, \alpha_2]$ there exists in $\mathbb{R}^2$ a unique line, which passes through a with a normal vector $\mathfrak{a}$. It is consisted of all points (x_1, x_2) satisfying the equation

$$\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 = 0, \text{ where } \alpha_0 = -a \cdot (\mathfrak{a}).$$

That is the linear equation of a line L such that $a \in L$ and $\mathfrak{a} \perp L$.

Proof. Let $a = (a_1, a_2) \in L$, $\mathfrak{a} = [\alpha_1, \alpha_2] \perp L$ and $b = (x_1, x_2) \in \mathbb{R}^2$. Then (see the picture)

$$\begin{aligned} b \in L \Leftrightarrow \left[\overrightarrow{ab}\right] \perp \mathfrak{a} &\Leftrightarrow \left[\overrightarrow{ab}\right] \cdot \mathfrak{a} = 0 \\ &\Leftrightarrow [x_1 - a_1, x_2 - a_2] \cdot [\alpha_1, \alpha_2] = 0 \\ &\Leftrightarrow \alpha_1(x_1 - a_1) + \alpha_2(x_2 - a_2) = 0 \\ &\Leftrightarrow -(a_1\alpha_1 + a_2\alpha_2) + \alpha_1 x_1 + \alpha_2 x_2 = 0. \end{aligned}$$



Setting

$$\alpha_0 = -(a_1\alpha_1 + a_2\alpha_2) = -a \cdot (\mathfrak{a})$$

we get

$$L : \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 = 0.$$

Obviously, such line is unique. □

Theorem. Let $L, K \subseteq \mathbb{R}^2$ be lines, $L : \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 = 0$ and $K : \beta_0 + \beta_1 x_1 + \beta_2 x_2 = 0$. Then

$$K = L \;\Leftrightarrow\; \bigvee_{t \neq 0} \beta_i = t\alpha_i \;\text{ for }\; i = 0, 1, 2.$$

$$K \parallel L \;\Leftrightarrow\; \bigvee_{t \neq 0} \beta_i = t\alpha_i \;\text{ for }\; i = 1, 2.$$

Proof. Easy. □

Definition. Let $L, K \subseteq \mathbb{R}^2$ be lines and $a \in \mathbb{R}^2$. Then

$$\rho(a, L) \underset{df}{=} \rho(a, b), \text{ where } b \in K \cap L \text{ and } a \in K \perp L$$

(a distance of a point a and a line L in $\mathbb{R}^2$).

Theorem. Let $L \subseteq \mathbb{R}^2$ be a line, $L : \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 = 0$ and $a = (a_1, a_2) \in \mathbb{R}^2$. Then

$$\rho(a, L) = \frac{|\alpha_0 + \alpha_1 a_1 + \alpha_2 a_2|}{\sqrt{\alpha_1^2 + \alpha_2^2}}.$$

Proof. Let $\mathfrak{a} = [\alpha_1, \alpha_2] \perp L$ and $x = (x_1, x_2) \in \mathbb{R}^2$. Then $L : \alpha_0 + x \cdot (\mathfrak{a}) = 0$. Take a line K such that $K : x(t) = a + t\mathfrak{a}$. Then $b \in K \cap L$, that is, $b = a + t'\mathfrak{a}$ and $\alpha_0 + b \cdot (\mathfrak{a}) = 0$, whence

$$\begin{aligned} \alpha_0 + (a + t'\mathfrak{a}) \cdot (\mathfrak{a}) &= 0 \\ \alpha_0 + a \cdot (\mathfrak{a}) + t'\mathfrak{a}^2 &= 0 \\ t'\mathfrak{a}^2 &= -\alpha_0 - a \cdot (\mathfrak{a}) \\ t' &= -\frac{\alpha_0 + a \cdot (\mathfrak{a})}{\mathfrak{a}^2}. \end{aligned}$$

Thus $b = a - \frac{\alpha_0 + a \cdot (\mathfrak{a})}{\mathfrak{a}^2}\mathfrak{a}$ and

$$\begin{aligned} \rho(a, L) = \rho(a, b) &= |b - a| \\ &= \left| a - \frac{\alpha_0 + a \cdot (\mathfrak{a})}{\mathfrak{a}^2}\mathfrak{a} - a \right| \\ &= \frac{|\alpha_0 + a \cdot (\mathfrak{a})|}{|\mathfrak{a}|^2} |\mathfrak{a}| \\ &= \frac{|\alpha_0 + \alpha_1 a_1 + \alpha_2 a_2|}{\sqrt{\alpha_1^2 + \alpha_2^2}}. \quad \square \end{aligned}$$

Definition. An equation $\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 = 0$ of a line L in $\mathbb{R}^2$ is called *normalized* if $\mathfrak{a} = [\alpha_1, \alpha_2]$ is a versor (so $|\mathfrak{a}| = 1$).

Conclusion. If $\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 = 0$ is a normalized equation of a line L in $\mathbb{R}^2$ and $a = (a_1, a_2) \in \mathbb{R}^2$, then

$$\rho(a, L) = |\alpha_0 + \alpha_1 a_1 + \alpha_2 a_2|.$$

Theorem. Every line in $\mathbb{R}^2$ has a normalized equation.

Proof. Easy. $\square$

Theorem. Let $L(a, b) \subseteq \mathbb{R}^2$ be a line, $a = (a_1, a_2), b = (b_1, b_2) \in \mathbb{R}^2$ and $a \neq b$. Then

$$L(a, b) : \begin{vmatrix} 1 & a_1 & a_2 \\ 1 & b_1 & b_2 \\ 1 & x_1 & x_2 \end{vmatrix} = 0.$$

Proof. We have $\left[\overrightarrow{ab}\right] = [b_1 - a_1, b_2 - a_2] \parallel L(a, b)$. It is easy to see that

$$[b_1 - a_1, b_2 - a_2] \cdot [-(b_2 - a_2), b_1 - a_1] = 0,$$

whence

$$[-(b_2 - a_2), b_1 - a_1] \perp L(a,b)$$

so

$$L(a,b) : -(a_1, a_2) \cdot (-(b_2 - a_2), b_1 - a_1) - (b_2 - a_2)x_1 + (b_1 - a_1)x_2 = 0.$$

Hence

$$L(a,b) : (a_2 x_1 + b_1 x_2 + a_1 b_2) - (b_2 x_1 + a_1 x_2 + a_2 b_1) = 0,$$

that is,

$$L(a,b) : \begin{vmatrix} 1 & a_1 & a_2 \\ 1 & b_1 & b_2 \\ 1 & x_1 & x_2 \end{vmatrix} = 0. \quad \square$$

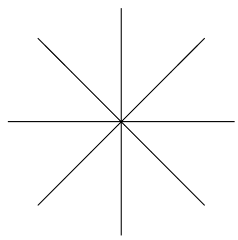
Remark. Niech L, K be lines in $\mathbb{R}^2$. Then

$L \parallel K \;\Rightarrow\; L = K \;\vee\; L \cap K = \emptyset,$

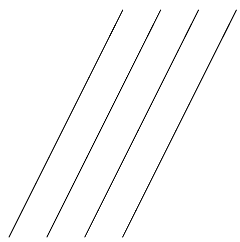
$L \nparallel K \;\Rightarrow\; L \cap K$ is a point.

Definition.

A *proper pencil of lines* in $\mathbb{R}^2 \underset{df}{=}$ the set of all lines which pass through one point



An *improper pencil of lines* in $\mathbb{R}^2 \underset{df}{=}$ the set of all lines with the same direction



Remark. Every two different lines in $\mathbb{R}^2$ determine a pencil (proper or improper). We use the following denotation:

$\mathrm{P}(L, K)$ = a pencil of lines in $\mathbb{R}^2$ determined by lines L, K.

Theorem. (**On a pencil of lines in** $\mathbb{R}^2$) Let $L : \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 = 0$, $K : \beta_0 + \beta_1 x_1 + \beta_2 x_2 = 0$ and $L \neq K$. Then

$$M \in \mathrm{P}(L, K) \Leftrightarrow \bigvee_{\eta, \lambda \in \mathbb{R},\, \eta^2 + \lambda^2 > 0} M : \eta(\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2) + \lambda(\beta_0 + \beta_1 x_1 + \beta_2 x_2) = 0.$$

Proof. First, note that if $\eta^2 + \lambda^2 > 0$, then an equation

$$\eta(\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2) + \lambda(\beta_0 + \beta_1 x_1 + \beta_2 x_2) = 0$$

is a linear equation of some line in $\mathbb{R}^2$. Indeed, we have $[\alpha_1, \alpha_2] \neq 0 \neq [\beta_1, \beta_2]$, whence $[\eta\alpha_1 + \lambda\beta_1, \eta\alpha_2 + \lambda\beta_2] = \eta[\alpha_1, \alpha_2] + \lambda[\beta_1, \beta_2] \neq 0$.

($\Rightarrow$) Let $M \in \mathrm{P}(L, K)$, $a = (a_1, a_2) \in M$ and $a \notin L \cup K$. It suffices to set: $\eta = \beta_0 + \beta_1 a_1 + \beta_2 a_2$ and $\lambda = -(\alpha_0 + \alpha_1 a_1 + \alpha_2 a_2)$.

($\Leftarrow$) Assume that

$$\bigvee_{\eta,\lambda\in\mathbb{R},\, \eta^2+\lambda^2>0} M : \eta(\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2) + \lambda(\beta_0 + \beta_1 x_1 + \beta_2 x_2) = 0.$$

We have two cases:

1) $\mathrm{P}(L, K)$ is proper.
Then an intersection point of lines L and K satisfies the equation of a line M, that is, $M \in \mathrm{P}(L, K)$.

2) $\mathrm{P}(L, K)$ is improper.
Then $\bigvee_{t\neq 0} [\beta_1, \beta_2] = t[\alpha_1, \alpha_2]$ (they are parallel), whence

$$\begin{aligned}[\eta\alpha_1 + \lambda\beta_1, \eta\alpha_2 + \lambda\beta_2] &= \eta[\alpha_1, \alpha_2] + \lambda[\beta_1, \beta_2] \\ &= \eta[\alpha_1, \alpha_2] + \lambda t[\alpha_1, \alpha_2] \\ &= (\eta + \lambda t)[\alpha_1, \alpha_2],\end{aligned}$$

that is, $M \parallel L \parallel K$. $\square$

Remark. Equivalently, we have

$$M \in \mathrm{P}(L, K) \Leftrightarrow \bigvee_{\lambda\in\mathbb{R}} M : \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \lambda(\beta_0 + \beta_1 x_1 + \beta_2 x_2) = 0$$

(in this case there does not exist λ such that $M = K$).

Definition.

Copenciled lines in $\mathbb{R}^2 \underset{df}{=}$ lines which belong to one pencil.

Theorem. Let $L : \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 = 0$, $K : \beta_0 + \beta_1 x_1 + \beta_2 x_2 = 0$ and $M : \gamma_0 + \gamma_1 x_1 + \gamma_2 x_2 = 0$ be distinct lines. Then lines L, K, M are copenciled $\Leftrightarrow$

$$\begin{vmatrix} \alpha_0 & \beta_0 & \gamma_0 \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{vmatrix} = 0.$$

Proof. We have $M \in \mathrm{P}(L, K) \Leftrightarrow$ there are $\eta, \lambda, \delta \in \mathbb{R},\ \eta^2 + \lambda^2 > 0$ such that

$$\begin{cases} \eta\alpha_0 + \lambda\beta_0 = -\delta\gamma_0, \\ \eta\alpha_1 + \lambda\beta_1 = -\delta\gamma_1, \\ \eta\alpha_2 + \lambda\beta_2 = -\delta\gamma_2, \end{cases}$$

which is equivalent to

$$\begin{cases} \eta\alpha_0 + \lambda\beta_0 + \delta\gamma_0 = 0, \\ \eta\alpha_1 + \lambda\beta_1 + \delta\gamma_1 = 0, \\ \eta\alpha_2 + \lambda\beta_2 + \delta\gamma_2 = 0. \end{cases}$$

That system has a nonzero solution $\Leftrightarrow$

$$\begin{vmatrix} \alpha_0 & \beta_0 & \gamma_0 \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{vmatrix} = 0. \qquad \square$$

Definition.

A *plane* $\underset{df}{=}$ a subspace of the space $\mathbb{R}^n$ isometric with $\mathbb{R}^2$.

Definition. Let $\mathfrak{a}, \mathfrak{b}, a, b \in \mathbb{R}^n$ and let $P \subseteq \mathbb{R}^n$ be a plane. Then

$\overrightarrow{ab}$ lies on $P \underset{df}{\Leftrightarrow} a, b \in P$.

$\mathfrak{a} \parallel P \underset{df}{\Leftrightarrow} \bigvee_{\overrightarrow{ab}} \overrightarrow{ab} \in \mathfrak{a} \ \wedge\ \overrightarrow{ab}$ lies on $P \ \Leftrightarrow \bigvee_{a,b \in P} \overrightarrow{ab} \in \mathfrak{a}$.

$\mathfrak{b} \perp P \underset{df}{\Leftrightarrow} \bigwedge_{\mathfrak{a} \parallel P} \mathfrak{b} \perp \mathfrak{a}$.

Definition. Let $P \subseteq \mathbb{R}^3$ be a plane and $\mathfrak{a} \in \mathbb{R}^3$.

A *normal direction* of a plane $P \underset{df}{=}$ a direction of a vector $\mathfrak{a} \perp P$.

A *normal vector* of a plane $P \underset{df}{=}$ a vector $\mathfrak{a} \perp P$.

Definition. Let $P, Q \subseteq \mathbb{R}^3$ be planes and $\mathfrak{a}, \mathfrak{b} \in \mathbb{R}^3$. Then

$$P \parallel Q \underset{df}{\Leftrightarrow} \mathfrak{a} \perp P \ \wedge\ \mathfrak{b} \perp Q \ \wedge\ \mathfrak{a} \parallel \mathfrak{b}.$$

$$P \perp Q \underset{df}{\Leftrightarrow} \mathfrak{a} \perp P \ \wedge\ \mathfrak{b} \perp Q \ \wedge\ \mathfrak{a} \perp \mathfrak{b}.$$

Theorem. For every point $a \in \mathbb{R}^3$ and every nonzero vector $\mathfrak{a} = [\alpha_1, \alpha_2, \alpha_3]$ there exists in $\mathbb{R}^3$ a unique plane, which passes through a with a normal vector $\mathfrak{a}$. It is consisted of all points (x_1, x_2, x_3) satisfying the equation

$$\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0, \text{ where } \alpha_0 = -a \cdot (\mathfrak{a}).$$

That is the linear equation of a plane P such that $a \in P$ and $\mathfrak{a} \perp P$.

Proof. Similar to the proof of theorem on a linear equation of a line. $\square$

Theorem. Let $P, Q \subseteq \mathbb{R}^3$ be planes, $P : \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0$ and $Q : \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 = 0$. Then

$$P = Q \;\Leftrightarrow\; \bigvee_{t \neq 0} \beta_i = t\alpha_i \;\text{ for }\; i = 0, 1, 2, 3.$$

$$P \parallel Q \;\Leftrightarrow\; \bigvee_{t \neq 0} \beta_i = t\alpha_i \;\text{ for }\; i = 1, 2, 3.$$

Proof. Easy. □

Definition. Let $P \subseteq \mathbb{R}^3$ be a plane, $L \subseteq \mathbb{R}^3$ be a line and $a \in \mathbb{R}^3$. Then

$$\rho(a, P) \underset{df}{=} \rho(a, b), \text{ where } b \in P \cap L \text{ and } a \in L \perp P$$

(a distance of a point a and a plane P in $\mathbb{R}^3$).

Theorem. Let $P \subseteq \mathbb{R}^3$ be a plane, $P : \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0$ and $a = (a_1, a_2, a_3) \in \mathbb{R}^3$. Then

$$\rho(a, P) = \frac{|\alpha_0 + \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3|}{\sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}}.$$

Proof. Similar to the proof of appropriate theorem for a line. □

Definition. An equation $\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0$ of a plane P in $\mathbb{R}^3$ is called *normalized* if $\mathfrak{a} = [\alpha_1, \alpha_2, \alpha_3]$ is a versor (so $|\mathfrak{a}| = 1$).

Conclusion. If $\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0$ is a normalized equation of a plane P in $\mathbb{R}^3$ and $a = (a_1, a_2, a_3) \in \mathbb{R}^3$, then

$$\rho(a, P) = |\alpha_0 + \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3|\,.$$

Theorem. Every plane in $\mathbb{R}^3$ has a normalized equation.

Proof. Easy. □

Theorem. Let $P \subseteq \mathbb{R}^3$ be a plane, $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3), c = (c_1, c_2, c_3) \in \mathbb{R}^3$ and $\overrightarrow{ab} \nparallel \overrightarrow{ac}$. Then

$$P : \begin{vmatrix} 1 & a_1 & a_2 & a_3 \\ 1 & b_1 & b_2 & b_3 \\ 1 & c_1 & c_2 & c_3 \\ 1 & x_1 & x_2 & x_3 \end{vmatrix} = 0.$$

Proof. Analogous to that for a line in $\mathbb{R}^2$. □

Remark. Let $P, Q \subseteq \mathbb{R}^3$ be planes. Then

$$P \parallel Q \;\Rightarrow\; P = Q \;\vee\; P \cap Q = \emptyset,$$

$$P \nparallel Q \;\Rightarrow\; P \cap Q \text{ is a line.}$$

Definition.

A *proper pencil* of planes in $\mathbb{R}^3 \underset{df}{=}$ the set of all planes containing the same line.

An *improper pencil* of planes in $\mathbb{R}^3 \underset{df}{=}$ the set of all planes with the same normal direction.

Remark. Every two different planes in $\mathbb{R}^3$ determine a pencil (proper or improper). We use the following denotation:

$\mathrm{P}(P, Q) =$ a pencil of planes in $\mathbb{R}^3$ determined by planes P, Q.

Theorem. (**On a pencil of planes in** $\mathbb{R}^3$) Let $P : \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0$, $Q : \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 = 0$ and $P \neq Q$. Then

$$R \in \mathrm{P}(P, Q) \Leftrightarrow \bigvee_{\eta,\lambda\in\mathbb{R},\, \eta^2+\lambda^2>0} R : \eta(\alpha_0+\alpha_1x_1+\alpha_2x_2+\alpha_3x_3)+\lambda(\beta_0+\beta_1x_1+\beta_2x_2+\beta_3x_3) = 0.$$

Proof. Analogous to that for a pencil of lines in $\mathbb{R}^2$. $\square$

Remark. Equivalently, we have

$$R \in \mathrm{P}(P, Q) \Leftrightarrow \bigvee_{\lambda\in\mathbb{R}} R : \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \lambda(\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3) = 0$$

(in this case there does not exist λ such that $R = Q$).

Remark. Let $P, Q \subseteq \mathbb{R}^3$ be planes and $P \nparallel Q$. Then $P \cap Q = L$ is a line. If $P : \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0$, $Q : \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 = 0$, then

$$L : \begin{cases} \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0, \\ \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 = 0. \end{cases}$$

It is an *edge equation* of a line L in $\mathbb{R}^3$. Then $\mathfrak{a} = [\alpha_1, \alpha_2, \alpha_3] \perp L$ and $\mathfrak{b} = [\beta_1, \beta_2, \beta_3] \perp L$. Hence $\mathfrak{a} \times \mathfrak{b} \parallel L$.

Definition. Let $L \subseteq \mathbb{R}^3$ be a line, $P \subseteq \mathbb{R}^3$ be a plane and $a \in \mathbb{R}^3$. Then

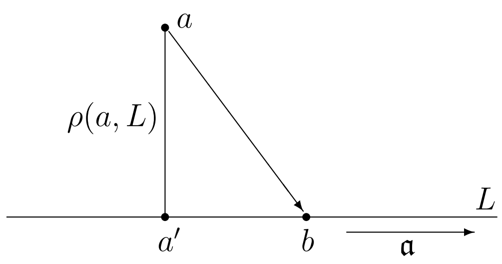
$$\rho(a, L) \underset{df}{=} \rho(a, b), \text{ where } b \in L \cap P \text{ and } a \in P \perp L$$

(a distance of a point a and a line L in $\mathbb{R}^3$).

Theorem. Let $L \subseteq \mathbb{R}^3$ be a line, $\mathfrak{a}, a, b \in \mathbb{R}^3$, $\mathfrak{a} \parallel L$, $a \neq b$ and $b \in L$. Then

$$\rho(a, L) = \frac{\left|\mathfrak{a} \times \left[\overrightarrow{ab}\right]\right|}{|\mathfrak{a}|}.$$

Proof. We have



Hence $\sin\left(\sphericalangle\left(\mathfrak{a}, \left[\overrightarrow{ab}\right]\right)\right) = \frac{\rho(a,a')}{\rho(a,b)}$ and

$$\rho(a, L) = \rho(a, a') = \rho(a, b) \sin\left(\sphericalangle\left(\mathfrak{a}, \left[\overrightarrow{ab}\right]\right)\right)$$
$$= \frac{|\mathfrak{a}|\left|\left[\overrightarrow{ab}\right]\right| \sin\left(\sphericalangle\left(\mathfrak{a}, \left[\overrightarrow{ab}\right]\right)\right)}{|\mathfrak{a}|}$$
$$= \frac{\left|\mathfrak{a} \times \left[\overrightarrow{ab}\right]\right|}{|\mathfrak{a}|}. \quad \square$$

Definition. Let $k < n$.

A *k-dimensional hyperplane* in $\mathbb{R}^n \underset{df}{=}$ a subspace of the space $\mathbb{R}^n$ isometric with $\mathbb{R}^k$.

Definition. Let $\mathfrak{a}, \mathfrak{b}, a, b \in \mathbb{R}^n$, H^{n-1} be an $(n-1)$-dimensional hyperplane in $\mathbb{R}^n$. Then

$$\mathfrak{a} \parallel H^{n-1} \underset{df}{\Leftrightarrow} \bigvee_{\overrightarrow{ab}} \overrightarrow{ab} \in \mathfrak{a} \ \wedge \ a, b \in H^{n-1} \ \Leftrightarrow \bigvee_{a,b \in H^{n-1}} \overrightarrow{ab} \in \mathfrak{a}.$$

$$\mathfrak{b} \perp H^{n-1} \underset{df}{\Leftrightarrow} \bigwedge_{\mathfrak{a} \parallel H^{n-1}} \mathfrak{b} \perp \mathfrak{a}.$$

Theorem. For every point $a \in \mathbb{R}^n$ and every nonzero vector $\mathfrak{a} = [\alpha_1, \ldots, \alpha_n]$ there exists in $\mathbb{R}^n$ a unique hyperplane H^{n-1} such that $a \in H^{n-1}$ and $\mathfrak{a} \perp H^{n-1}$. It is consisted of all points $(x_1, \ldots, x_n)$ satisfying the equation

$$\alpha_0 + \alpha_1 x_1 + \ldots + \alpha_n x_n = 0, \text{ where } \alpha_0 = -a \cdot (\mathfrak{a}).$$

That is the linear equation of a hyperplane H^{n-1} such that $a \in H^{n-1}$ and $\mathfrak{a} \perp H^{n-1}$.

Proof. Similar to the proof of theorem on a linear equation of a line. $\square$

5. Transformations of space $\mathbb{R}^n$

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be an isometry, that is, f is onto and

$$\bigwedge_{x,y \in \mathbb{R}^n} \rho(f(x), f(y)) = \rho(x, y).$$

Definition.

An *invariant of isometry* $\underset{df}{=}$ a property which is unchanged by isometries.

Theorem. A centre of a segment is an invariant of isometry (that is, if c is a centre of a segment $\langle a, b\rangle$, then $f(c)$ is a centre of a segment $\langle f(a), f(b)\rangle$).

Proof. Take an isometry $f : \mathbb{R}^n \to \mathbb{R}^n$ and let $a, b, c \in \mathbb{R}^n$. If c is a centre of a segment $\langle a, b\rangle$, then

$$\rho(a, c) = \rho(b, c) = \frac{1}{2}\rho(a, b).$$

Hence

$$\rho(f(a), f(c)) = \rho(f(b), f(c)) = \frac{1}{2}\rho(f(a), f(b)),$$

that is, $f(c)$ is a centre of a segment $\langle f(a), f(b)\rangle$. □

Theorem. An equality of localized vectors is an invariant of isometry.

Proof. Follows from definition of equal vectors and previous theorem. □

Conclusion. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be an isometry and $\mathfrak{a}, a, b \in \mathbb{R}^n$. Then

$$\mathfrak{a} = \left[\overrightarrow{ab}\right] \;\Rightarrow\; f(\mathfrak{a}) = \left[\overrightarrow{f(a)f(b)}\right].$$

Theorem. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be an isometry and $\mathfrak{a}, \mathfrak{b} \in \mathbb{R}^n$. Then

1) $f(0) = 0$ (for vectors!),
2) $f(\mathfrak{a} + \mathfrak{b}) = f(\mathfrak{a}) + f(\mathfrak{b})$,
3) $f(-\mathfrak{a}) = -f(\mathfrak{a})$,
4) $f(\mathfrak{a} - \mathfrak{b}) = f(\mathfrak{a}) - f(\mathfrak{b})$,
5) $|f(\mathfrak{a})| = |\mathfrak{a}|$.

Proof. 1) Obvious.

2) Let $\mathfrak{a}, \mathfrak{b}, a, b, c \in \mathbb{R}^n$. Then $\overrightarrow{ab} \in \mathfrak{a}$ and $\overrightarrow{bc} \in \mathfrak{b}$ from theorem on localization of a free vector at a point. Thus $\overrightarrow{ab} + \overrightarrow{bc} = \overrightarrow{ac} \in \mathfrak{a} + \mathfrak{b}$. Hence $\overrightarrow{f(a)f(c)} \in f(\mathfrak{a} + \mathfrak{b})$ and $\overrightarrow{f(a)f(c)} = \overrightarrow{f(a)f(b)} + \overrightarrow{f(b)f(c)} \in f(\mathfrak{a}) + f(\mathfrak{b})$. Thus $f(\mathfrak{a} + \mathfrak{b}) = f(\mathfrak{a}) + f(\mathfrak{b})$.

3) We have

$$0 = f(0) = f(\mathfrak{a} + (-\mathfrak{a})) = f(\mathfrak{a}) + f(-\mathfrak{a}).$$

Hence $f(-\mathfrak{a}) = -f(\mathfrak{a})$.

4) We easily get

$$f(\mathfrak{a}-\mathfrak{b}) = f(\mathfrak{a}+(-\mathfrak{b})) = f(\mathfrak{a}) + f(-\mathfrak{b}) = f(\mathfrak{a}) - f(\mathfrak{b}).$$

5) Take $a, b \in \mathbb{R}^n$ such that $\overrightarrow{ab} \in \mathfrak{a}$. We have

$$|f(\mathfrak{a})| = \left|\left[\overrightarrow{f(a)f(b)}\right]\right| = \rho(f(a), f(b)) = \rho(a, b) = \left|\left[\overrightarrow{ab}\right]\right| = |\mathfrak{a}|. \quad \square$$

Conclusion. The zero vector, an opposite vector, a sum and a difference of vectors and a length of a vector are invariants of isometry.

Theorem. Parallelism, equally parallelism and oppositely parallelism of vectors are invariants of isometry.

Proof. Follows from definition of parallelism and previous theorem. $\square$

Conclusion. A direction and a sense of a vector are invariants of isometry, that is, for $\mathfrak{a} \in \mathbb{R}^n$,

$$f(\mathcal{K}(\mathfrak{a})) = \mathcal{K}(f(\mathfrak{a})) \quad \text{and}$$
$$f(\mathcal{Z}(\mathfrak{a})) = \mathcal{Z}(f(\mathfrak{a})).$$

Theorem. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be an isometry, $\mathfrak{a} \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Then

$$f(t\mathfrak{a}) = tf(\mathfrak{a}).$$

Proof. Assume $t \geq 0$. Then $t\mathfrak{a} \upharpoonleft\upharpoonright \mathfrak{a}$, whence $f(t\mathfrak{a}) \upharpoonleft\upharpoonright f(\mathfrak{a})$ and $tf(\mathfrak{a}) \upharpoonleft\upharpoonright f(\mathfrak{a})$. Thus

$$f(t\mathfrak{a}) \upharpoonleft\upharpoonright tf(\mathfrak{a}).$$

Moreover,

$$|f(t\mathfrak{a})| = |t\mathfrak{a}| = t\,|\mathfrak{a}| = t\,|f(\mathfrak{a})|\,.$$

Hence $f(t\mathfrak{a}) = tf(\mathfrak{a})$.

Similarly for $t < 0$ (in that case parallelism is opposite). $\square$

Conclusion. A linear combination of vectors is an invariant of isometry, that is,

$$f\left(\sum_{i=1}^{k} t_i\mathfrak{a}_i\right) = \sum_{i=1}^{k} t_i f(\mathfrak{a}_i),$$

where $\mathfrak{a}_1, \ldots, \mathfrak{a}_k \in \mathbb{R}^n$ and $t_1, \ldots, t_k \in \mathbb{R}$.

Theorem. A scalar product of vectors is an invariant of isometry, that is,

$$f(\mathfrak{a}) \cdot f(\mathfrak{b}) = \mathfrak{a} \cdot \mathfrak{b}.$$

Proof. Take an isometry $f:\mathbb{R}^n\to\mathbb{R}^n$ and let $\mathfrak{a},\mathfrak{b}\in\mathbb{R}^n$. We have

$$(f(\mathfrak{a})+f(\mathfrak{b}))^2=(f(\mathfrak{a}+\mathfrak{b}))^2=|f(\mathfrak{a}+\mathfrak{b})|^2=|\mathfrak{a}+\mathfrak{b}|^2=(\mathfrak{a}+\mathfrak{b})^2=\mathfrak{a}^2+2\mathfrak{a}\cdot\mathfrak{b}+\mathfrak{b}^2=|\mathfrak{a}|^2+2\mathfrak{a}\cdot\mathfrak{b}+|\mathfrak{b}|^2$$

and

$$(f(\mathfrak{a})+f(\mathfrak{b}))^2=(f(\mathfrak{a}))^2+2f(\mathfrak{a})\cdot f(\mathfrak{b})+(f(\mathfrak{b}))^2=|f(\mathfrak{a})|^2+2f(\mathfrak{a})\cdot f(\mathfrak{b})+|f(\mathfrak{b})|^2=|\mathfrak{a}|^2+2f(\mathfrak{a})\cdot f(\mathfrak{b})+|\mathfrak{b}|^2.$$

Hence $|\mathfrak{a}|^2+2\mathfrak{a}\cdot\mathfrak{b}+|\mathfrak{b}|^2=|\mathfrak{a}|^2+2f(\mathfrak{a})\cdot f(\mathfrak{b})+|\mathfrak{b}|^2$. Thus

$$f(\mathfrak{a})\cdot f(\mathfrak{b})=\mathfrak{a}\cdot\mathfrak{b}.\quad\square$$

Conclusion. A perpendicularity of vectors is an invariant of isometry.

Conclusion. A cosine of an angle between vectors and a measure of an angle between vectors are invariants of isometry.

Theorem. A k-dimensional hyperplane in $\mathbb{R}^n$ $(k<n)$ is an invariant of isometry, that is, if H^k is a k-dimensional hyperplane, then $f(H^k)$ is a k-dimensional hyperplane.

Proof. Follows from definition of a k-dimensional hyperplane and the fact that a composition of isometries is an isometry. $\square$

Conclusion. A line and a plane in $\mathbb{R}^n$ are invariants of isometry.

Conclusion. A pencil of lines in $\mathbb{R}^2$ and a pencil of planes in $\mathbb{R}^3$ are invariants of isometry.

Theorem. Parallelism and perpendicularity of lines in $\mathbb{R}^n$ and parallelism and perpendicularity of planes in $\mathbb{R}^3$ are invariants of isometry.

Proof. Follows from the fact that parallelism and perpendicularity of vectors are invariants of isometry. $\square$

Remark. Let us set:

$$\delta^i_j=\begin{cases}0 & \text{if } i\neq j,\\ 1 & \text{if } i=j.\end{cases}$$

Theorem. (**On an analytic form of an isometry**) Every isometry $f:\mathbb{R}^n\to\mathbb{R}^n$ is a transformation given by a formula

$$f(x)=a+\sum_{i=1}^{n}x_i\cdot(\mathfrak{a}_i),\ \text{where } \mathfrak{a}_i\cdot\mathfrak{a}_j=\delta^i_j.$$

Then $f(0)=a$ and $\mathfrak{a}_i=f(\mathfrak{e}_i)$, where $\mathfrak{e}_i=[\delta^i_1,\delta^i_2,\ldots,\delta^i_n]$, $i=1,\ldots,n$.

Proof. First, note that $\mathfrak{e}_1=[1,0,0,\ldots,0],\mathfrak{e}_2=[0,1,0,\ldots,0],\ldots,\mathfrak{e}_n=[0,0,0,\ldots,1]$. From properties of an isometry we know that an isometry is a linear transformation. Hence every isometry $f:\mathbb{R}^n\to\mathbb{R}^n$ is uniquely determined by its values $f(\mathfrak{e}_1),\ldots,f(\mathfrak{e}_n)$ in end-points of vectors $\mathfrak{e}_1,\ldots,\mathfrak{e}_n$.

Now, if $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, then $x = 0 + x_1\mathfrak{e}_1 + \ldots + x_n\mathfrak{e}_n$, whence $f(x) = f(0) + x_1 f(\mathfrak{e}_1) + \ldots + x_n f(\mathfrak{e}_n)$. Setting $f(0) = a$ and $f(\mathfrak{e}_i) = \mathfrak{a}_i$, $i = 1, \ldots, n$ we get $\mathfrak{a}_i \cdot \mathfrak{a}_j = \delta^i_j$ and

$$f(x) = a + \sum_{i=1}^{n} x_i \cdot (\mathfrak{a}_i). \quad \square$$

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a similarity with the ratio $\lambda > 0$, that is, f is onto and

$$\bigwedge_{x,y \in \mathbb{R}^n} \rho(f(x), f(y)) = \lambda \rho(x, y).$$

Definition.

A *similarity invariant* $\underset{df}{=}$ a property which is unchanged by similarities.

Remark. Every similarity invariant is an invariant of isometry (since an isometry is a similarity with the ratio 1). An invariant of isometry is a similarity invariant iff it does not depend on a distance of points in $\mathbb{R}^n$. Thus we have:

Theorem. Similarity invariants are: a centre of a segment, an equality of localized vectors, the zero vector, an opposite vector, a sum and a difference of vectors, a parallelism, an equally parallelism and an oppositely parallelism of vectors, a direction and a sense of a vector, a linear combination of vectors, a k-dimensional hyperplane in $\mathbb{R}^n$, a line in $\mathbb{R}^n$, a plane in $\mathbb{R}^n$, a parallelism and a perpendicularity of lines in $\mathbb{R}^n$ and a parallelism and a perpendicularity of planes in $\mathbb{R}^3$, a pencil of lines in $\mathbb{R}^2$ and a pencil of planes in $\mathbb{R}^3$.

Conclusion. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a similarity and $\mathfrak{a}, a, b \in \mathbb{R}^n$. Then

$$\mathfrak{a} = \left[\overrightarrow{ab}\right] \;\Rightarrow\; f(\mathfrak{a}) = \left[\overrightarrow{f(a)f(b)}\right].$$

Theorem. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a similarity with the ratio $\lambda > 0$ and $\mathfrak{a}, \mathfrak{b} \in \mathbb{R}^n$. Then

1) $|f(\mathfrak{a})| = \lambda\,|\mathfrak{a}|$,
2) $f(\mathfrak{a}) \cdot f(\mathfrak{b}) = \lambda^2(\mathfrak{a} \cdot \mathfrak{b})$.

Proof. 1) Let $a, b \in \mathbb{R}^n$ be such that $\overrightarrow{ab} \in \mathfrak{a}$. We have

$$|f(\mathfrak{a})| = \left|\left[\overrightarrow{f(a)f(b)}\right]\right| = \rho(f(a), f(b)) = \lambda\rho(a, b) = \lambda\left|\left[\overrightarrow{ab}\right]\right| = \lambda\,|\mathfrak{a}|\,.$$

2) We know that $f(\mathfrak{a}) + f(\mathfrak{b}) = f(\mathfrak{a} + \mathfrak{b})$. Hence

$$(f(\mathfrak{a})+f(\mathfrak{b}))^2 = (f(\mathfrak{a}+\mathfrak{b}))^2 = |f(\mathfrak{a} + \mathfrak{b})|^2 = \lambda^2\,|\mathfrak{a} + \mathfrak{b}|^2 = \lambda^2(\mathfrak{a}+\mathfrak{b})^2 = \lambda^2\,|\mathfrak{a}|^2 + 2\lambda^2\mathfrak{a}\cdot\mathfrak{b} + \lambda^2\,|\mathfrak{b}|^2$$

and

$$(f(\mathfrak{a})+f(\mathfrak{b}))^2 = (f(\mathfrak{a}))^2 + 2f(\mathfrak{a})\cdot f(\mathfrak{b}) + (f(\mathfrak{b}))^2$$
$$= |f(\mathfrak{a})|^2 + 2f(\mathfrak{a})\cdot f(\mathfrak{b}) + |f(\mathfrak{b})|^2$$
$$= \lambda^2 |\mathfrak{a}|^2 + 2f(\mathfrak{a})\cdot f(\mathfrak{b}) + \lambda^2 |\mathfrak{b}|^2 .$$

Thus

$$f(\mathfrak{a})\cdot f(\mathfrak{b}) = \lambda^2 \mathfrak{a}\cdot\mathfrak{b}. \ \square$$

Conclusion. A length of a vector and a scalar product of vectors are not similarity invariants.

Theorem. A cosine of an angle between vectors is a similarity invariant.

Proof. Let $f:\mathbb{R}^n \to \mathbb{R}^n$ be a similarity with the ratio $\lambda > 0$ and $\mathfrak{a},\mathfrak{b} \in \mathbb{R}^n$. By previous theorem we have

$$f(\mathfrak{a})\cdot f(\mathfrak{b}) = |f(\mathfrak{a})|\,|f(\mathfrak{b})| \cos(\sphericalangle(f(\mathfrak{a}), f(\mathfrak{b}))) = \lambda^2 |\mathfrak{a}|\,|\mathfrak{b}| \cos(\sphericalangle(f(\mathfrak{a}), f(\mathfrak{b})))$$

and

$$\lambda^2(\mathfrak{a}\cdot\mathfrak{b}) = \lambda^2 |\mathfrak{a}|\,|\mathfrak{b}| \cos(\sphericalangle(\mathfrak{a},\mathfrak{b})),$$

that is,

$$\cos(\sphericalangle(f(\mathfrak{a}), f(\mathfrak{b}))) = \cos(\sphericalangle(\mathfrak{a},\mathfrak{b})). \ \square$$

Conclusion. A measure of an angle between vectors, in particular, a perpendicularity of vectors are similarity invariants.

Theorem. (**On an analytic form of a similarity**) Every similarity $f:\mathbb{R}^n \to \mathbb{R}^n$ with the ratio $\lambda > 0$ is a transformation given by a formula

$$f(x) = a + \sum_{i=1}^{n} x_i \cdot (\mathfrak{a}_i), \ \text{ where } \mathfrak{a}_i \cdot \mathfrak{a}_j = \lambda^2 \delta^i_j.$$

Then $f(0) = a$ and $\mathfrak{a}_i = f(\mathfrak{e}_i)$, where $\mathfrak{e}_i = [\delta^i_1, \delta^i_2, \ldots, \delta^i_n]$, $i = 1, \ldots, n$.

Proof. We have that $g:\mathbb{R}^n \to \mathbb{R}^n$ such that $g(x) = \frac{1}{\lambda} f(x)$, where $x \in \mathbb{R}^n$, is an isometry, because

$$\rho(g(x), g(y))^2 = [g(y) - g(x)]^2 = \frac{1}{\lambda^2}[f(y) - f(x)]^2 = \frac{1}{\lambda^2}\rho(f(x), f(y))^2 = \rho(x,y)^2,$$

that is, $\rho(g(x), g(y)) = \rho(x, y)$, where $x, y \in \mathbb{R}^n$.

By theorem on an analytic form of an isometry

$$g(x) = b + \sum_{i=1}^{n} x_i \cdot (\mathfrak{b}_i),$$

where $\mathfrak{b}_i \cdot \mathfrak{b}_j = \delta^i_j$, $g(0) = b$, $\mathfrak{b}_i = g(\mathfrak{e}_i)$ and $\mathfrak{e}_i = [\delta^i_1, \delta^i_2, \ldots, \delta^i_n]$. Hence

$$f(x) = \lambda g(x) = \lambda b + \sum_{i=1}^{n} x_i \cdot (\lambda \mathfrak{b}_i).$$

Setting $a = \lambda b$ and $\mathfrak{a}_i = \lambda \mathfrak{b}_i$, $i = 1, \ldots, n$ we get

$$f(x) = a + \sum_{i=1}^{n} x_i \cdot (\mathfrak{a}_i)$$

and

$$\begin{aligned} \mathfrak{a}_i \cdot \mathfrak{a}_j &= (\lambda \mathfrak{b}_i) \cdot (\lambda \mathfrak{b}_j) = \lambda^2 (\mathfrak{b}_i \cdot \mathfrak{b}_j) = \lambda^2 \delta^i_j, \\ f(0) &= \lambda g(0) = \lambda b = a, \\ \mathfrak{a}_i &= \lambda \mathfrak{b}_i = \lambda g(\mathfrak{e}_i) = f(\mathfrak{e}_i), \end{aligned}$$

where $\mathfrak{e}_i = [\delta^i_1, \delta^i_2, \ldots, \delta^i_n]$, $i = 1, \ldots, n$. $\square$

Definition. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a transformation. Then

$$f \text{ is an } affine\ transformation \underset{df}{\Leftrightarrow} 1)\ f : \mathbb{R}^n \underset{1-1}{\overset{onto}{\longrightarrow}} \mathbb{R}^n,$$

$$2) \bigwedge_{a,b,a',b' \in \mathbb{R}^n} \overrightarrow{ab} = \overrightarrow{a'b'} \Rightarrow \overrightarrow{f(a)f(b)} = \overrightarrow{f(a')f(b')},$$

$$3) \bigwedge_{\mathfrak{a}_1, \mathfrak{a}_2 \in \mathbb{R}^n} \bigwedge_{t_1, t_2 \in \mathbb{R}} f(t_1 \mathfrak{a}_1 + t_2 \mathfrak{a}_2) = t_1 f(\mathfrak{a}_1) + t_2 f(\mathfrak{a}_2).$$

Conclusion. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be an affine transformation and $\mathfrak{a}, a, b \in \mathbb{R}^n$. Then

$$\mathfrak{a} = \left[\overrightarrow{ab}\right] \Rightarrow f(\mathfrak{a}) = \left[\overrightarrow{f(a)f(b)}\right].$$

Conclusion. Every isometry and every similarity are affine transformations.

Definition. Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_k \in \mathbb{R}^n$ and $t_1, \ldots, t_k \in \mathbb{R}$.

Vectors $\mathfrak{a}_1, \ldots, \mathfrak{a}_k$ are *linearly independent* $\underset{df}{\Leftrightarrow}$

$$\sum_{i=1}^{k} t_i \mathfrak{a}_i = 0 \Rightarrow t_1 = t_2 = \ldots = t_k = 0.$$

Theorem. (**On an analytic form of an affine transformation**) Every affine transformation $f : \mathbb{R}^n \to \mathbb{R}^n$ is given by a formula

$$f(x) = a + \sum_{i=1}^{n} x_i \cdot (\mathfrak{a}_i),$$

where vectors $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ are linearly independent. Then $f(0) = a$ and $\mathfrak{a}_i = f(\mathfrak{e}_i)$, where $\mathfrak{e}_i = [\delta^i_1, \delta^i_2, \ldots, \delta^i_n]$, $i = 1, \ldots, n$.

Proof. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we have $x = 0 + x_1 \cdot (\mathfrak{e}_1) + \ldots + x_n \cdot (\mathfrak{e}_n)$. By definition of an affine transformation

$$f(x) = f(0) + x_1 \cdot f(\mathfrak{e}_1) + \ldots + x_n \cdot f(\mathfrak{e}_n).$$

Let us set: $f(0) = a$ and $f(\mathfrak{e}_i) = \mathfrak{a}_i$, $i = 1, \ldots, n$. Then

$$f(x) = a + \sum_{i=1}^{n} x_i \cdot (\mathfrak{a}_i)$$

and from the fact that f is one-to-one:

$$f(x) = f(0) \ \Rightarrow \ x = 0,$$

$$\text{that is, } a + \sum_{i=1}^{n} x_i \cdot (\mathfrak{a}_i) = a \ \Rightarrow \ x_1 = \ldots = x_n = 0,$$

$$\text{so, } \sum_{i=1}^{n} x_i \cdot (\mathfrak{a}_i) = 0 \ \Rightarrow \ x_1 = \ldots = x_n = 0.$$

Hence vectors $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ are linearly independent. □

Theorem. Composition of two affine transformations is an affine transformation.

Proof. Easy. □

Theorem. If $f : \mathbb{R}^n \to \mathbb{R}^n$ is an affine transformation, then $f^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is an affine transformation.

Proof. Easy. □

Definition.

An *affine invariant* $\underset{df}{=}$ a property which is unchanged by affine transformations.

Conclusion. Affine invariants are: an equality of localized vectors, a linear combination of vectors and a parallelism of vectors.

Theorem. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be an affine transformation, $a, b \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Then

$$f((1-t)a + tb) = (1-t)f(a) + tf(b).$$

Proof. Easy. It suffices to use an analytic form of an affine transformation. □

Conclusion. A centre of a segment is an affine invariant.

Conclusion. A line in $\mathbb{R}^n$ is an affine invariant.

Conclusion. A plane in $\mathbb{R}^n$ and a k-dimensional hyperplane in $\mathbb{R}^n$ are affine invariants (because they are unions of lines).

Conclusion. A parallelism of lines in $\mathbb{R}^n$ and a parallelism of planes in $\mathbb{R}^3$ are affine invariants.

Remark. Every affine invariant is a similarity invariant (which means that if a property is not a similarity invariant, then it is not an affine invariant).

Conclusion. A length of a vector and a scalar product of vectors are not affine invariants.

Conclusion. A cosine of an angle between vectors, a measure of an angle between vectors, in particular, a perpendicularity of vectors are not affine invariants.

Conclusion. Every affine invariant is a similarity invariant and every similarity invariant is an invariant of isometry.

Now we will give some characterizations of an isometry, a similarity and an affine transformation. First we need a notion of an orthogonal matrix.

Definition. Let A be a real square matrix of order n.

A matrix A is called *orthogonal* $\underset{df}{\Leftrightarrow}$ columns of A are versors in $\mathbb{R}^n$ perpendicular to each other.

Theorem. Let A be a real square matrix. The following are equivalent:

1) A is orthogonal,
2) $A^T A = I$,
3) $A^{-1} = A^T$.

Proof. Easy. □

Conclusion. Let A, B be orthogonal matrices of order n. Then

1) $\det(A) = \pm 1$,
2) A^T is orthogonal,
3) rows of A are versors in $\mathbb{R}^n$ perpendicular to each other,
4) A^{-1} is orthogonal,
5) AB is orthogonal.

Definition. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be an isometry (a similarity, an affine transformation), $a = (a_{01}, \ldots, a_{0n}), \mathfrak{a}_i = [\alpha_{i1}, \ldots, \alpha_{in}] \in \mathbb{R}^n$ for $i = 1, \ldots, n$, and let $(x_1, \ldots, x_n), (\overline{x}_1, \ldots, \overline{x}_n) \in \mathbb{R}^n$. Then

$$f(x) = a + \sum_{i=1}^{n} x_i \cdot (\mathfrak{a}_i),$$

that is,

$$f(x_1, \ldots, x_n) = (\overline{x}_1, \ldots, \overline{x}_n) = (a_{01}, \ldots, a_{0n}) + x_1(\alpha_{11}, \ldots, \alpha_{1n}) + \ldots + x_n(\alpha_{n1}, \ldots, \alpha_{nn}),$$

so

$$\begin{cases} \overline{x}_1 = a_{01} + \alpha_{11}x_1 + \ldots + \alpha_{n1}x_n, \\ \overline{x}_2 = a_{02} + \alpha_{12}x_1 + \ldots + \alpha_{n2}x_n, \\ \quad\vdots \\ \overline{x}_n = a_{0n} + \alpha_{1n}x_1 + \ldots + \alpha_{nn}x_n. \end{cases}$$

A matrix

$$A_f \underset{df}{=} \begin{bmatrix} \alpha_{11} & \ldots & \alpha_{n1} \\ \alpha_{12} & \ldots & \alpha_{n2} \\ \vdots & \ddots & \vdots \\ \alpha_{1n} & \ldots & \alpha_{nn} \end{bmatrix}$$

is called the *matrix of an isometry* (*a similarity*, *an affine transformation*) f.

Theorem. A transformation $f : \mathbb{R}^n \to \mathbb{R}^n$ given by the above analytic formula is:

1) an affine transformation $\Leftrightarrow A_f$ is nonsingular,
2) a similarity with the ratio $\lambda > 0 \Leftrightarrow \frac{1}{\lambda}A_f$ is orthogonal,
3) an isometry $\Leftrightarrow A_f$ is orthogonal.

Proof. Follows from theorems on an analytic forms of these transformations. □

6. Algebraic sets in space $\mathbb{R}^n$

Definition. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a function, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $i_1, \ldots, i_n \in \{0, \ldots, k\}$ and $k \in \mathbb{N} \cup \{0\}$. Then

φ is a *monomial in n variables* $\underset{df}{\Leftrightarrow}$ $\varphi(x) = \alpha_{i_1 \ldots i_n} x_1^{i_1} \ldots x_n^{i_n}$.

A *degree* of a monomial $\varphi \underset{df}{=} i_1 + \ldots + i_n$.

φ is a *polynomial in n variables* $\underset{df}{\Leftrightarrow}$ φ is a sum of monomials.

A *degree* of a polynomial $\varphi \underset{df}{=}$ the greatest of degrees of monomials occuring in a polynomial φ.

Example.
1. $\varphi(x) = 2x_1^2 x_2^3$ is the monomial of degree 5 in 2 variables.
2. $\varphi(x) = x_1^2 x_2 + 2x_2^2 x_3^2 - 3x_1 x_3 + 5x_1 - 4$ is the polynomial of degree 4 in 3 variables.

Definition. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a polynomial of degree k. An equation $\varphi(x) = 0$ is called the *algebraic equation* of degree k.

Definition. (**An algebraic set in** $\mathbb{R}^n$) Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a polynomial and $\varphi(x) = 0$ be an algebraic equation.

An *algebraic set* $\underset{df}{=}$ a set of solutions of an algebraic equation,

that is, if $F \subseteq \mathbb{R}^n$, then

F is an algebraic set $\underset{df}{\Leftrightarrow}$ [there is a polynomial $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that $x \in F \Leftrightarrow \varphi(x) = 0$].

We will write $F : \varphi(x) = 0$.

A *degree* of a set $F \underset{df}{=}$ the least of degrees of algebraic equations describing a set F.

We denote it by $\deg(F)$.

Remarks.
1. Algebraic sets of degree 0 in $\mathbb{R}^n$: $\emptyset$ and $\mathbb{R}^n$ (since if a polynomial φ is of degree 0, then an equation $\varphi(x) = 0$ is either contradictory or it is an identity).
2. Algebraic sets of degree 1 in $\mathbb{R}^n$: $(n-1)$-dimensional hyperplanes (if $H^{n-1} : \alpha_0 + \alpha_1 x_1 + \ldots + \alpha_n x_n = 0$, then $\varphi(x_1, \ldots, x_n) = \alpha_0 + \alpha_1 x_1 + \ldots + \alpha_n x_n = 0$ is an algebraic equation of degree 1).
3. Algebraic sets of degree 2 in $\mathbb{R}^1$: 2-point sets (since a polynomial of degree 2 in one variable has at most 2 roots).

4. Algebraic sets of degree k in $\mathbb{R}^1$: k-point sets (since a polynomial of degree k in one variable has at most k roots).

Conclusion. A line in $\mathbb{R}^2$ and a plane in $\mathbb{R}^3$ are algebraic sets of degree 1.

Theorem. (**On position of a line under an algebraic set of degree** k)
Let $L, F \subseteq \mathbb{R}^n$, L be a line and F be an algebraic set of degree k. Then

$$L \subseteq F \quad \vee \quad \overline{\overline{L \cap F}} \leq k.$$

Proof. Let $F : \varphi(x) = 0$, where φ is a polynomial of degree k. By theorem on a line: $L : x(t) = (1-t)a + tb$, where $t \in \mathbb{R}$ and $a, b \in L$, that is, $L : (x_1, \ldots, x_n) = a + (b-a)t$, where $t \in \mathbb{R}$ and $a, b \in L$. We search all $t \in \mathbb{R}$ satisfying the following system of equations

$$\begin{cases} (x_1, \ldots, x_n) = a + (b-a)t, \\ \varphi(x_1, \ldots, x_n) = 0. \end{cases}$$

It is not difficult to see that there are no such t or all $t \in \mathbb{R}$ satisfy that system or at most k numbers t satisfy that system. Hence

$$L \cap F = \emptyset \quad \vee \quad L \cap F = L \quad \vee \quad \overline{\overline{L \cap F}} \leq \overline{\overline{\{t_1, \ldots, t_k\}}}.$$

Thus

$$L \subseteq F \quad \vee \quad \overline{\overline{L \cap F}} \leq k. \ \square$$

Definition.

A *transcendental set* $\underset{df}{=}$ a subset of $\mathbb{R}^n$ which is not an algebraic set of any degree.

Conclusion. If for a set $F \subseteq \mathbb{R}^n$ there exists a line L such that $L \cap F$ is a proper infinite subset of L, then the set F is transcendental.

Example. The sinusoid is a transcendental set.

Theorem. An algebraic set and its degree are affine invariants.

Proof. Let $F : \varphi(x) = 0$ be an algebraic set of degree k and $f : \mathbb{R}^n \to \mathbb{R}^n$ be an affine transformation. Then we know that f^{-1} is also an affine transformation. If $f(x_1, \ldots, x_n) = (\overline{x}_1, \ldots, \overline{x}_n)$, then $f^{-1}(\overline{x}_1, \ldots, \overline{x}_n) = (x_1, \ldots, x_n)$. From an analytic form of an affine transformation f^{-1} we have formulas for $x_1, \ldots, x_n$. We set them to the equation $\varphi(x_1, \ldots, x_n) = 0$ and obtain an algebraic equation of degree k of an algebraic set $\overline{F}$, that is, $f(F) = \overline{F}$. $\square$

Conclusion. An algebraic set and its degree are similarity invariants and also invariants of isometry.

Conclusion. A transcendental set is an affine invariant (so also a similarity invariant and an invariant of isometry).

Definition. Let $a, a' \in \mathbb{R}^n$ and $H \subseteq \mathbb{R}^n$ be a hyperplane. Then

a, a' are *symmetric with respect to* $H \underset{df}{\Leftrightarrow}$

$$c = \frac{a + a'}{2} \in H \ \wedge \ \left[\overrightarrow{aa'}\right] \perp H.$$

Definition. Let $F, H \subseteq \mathbb{R}^n$, F be an algebraic set and H be a hyperplane. Then

H is a *hyperplane of symmetry of* $F \underset{df}{\Leftrightarrow}$

$$[a \in F \ \Rightarrow \ a' \in F, \text{ where } a' \text{ is symmetric to } a \text{ with respect to } H].$$

Remarks.
1. A 0-dimensional hyperplane of symmetry reduces to a point, called a *centre of symmetry* of a set F.
2. A 1-dimensional hyperplane of symmetry is a line, called an *axis of symmetry* of a set F.

Theorem. A centre of symmetry of an algebraic set is an affine invariant.

Proof. Follows directly from definition. □

Remark. An axis of symmetry of an algebraic set is not an affine invariant.

Algebraic sets of degree 2 in $\mathbb{R}^2$:

1. A 1-point set.

Let $a = (a_1, a_2), x = (x_1, x_2) \in \mathbb{R}^2$. Then

$$\{a\} : (x_1 - a_1)^2 + (x_2 - a_2)^2 = 0$$

and $\varphi(x) = (x_1 - a_1)^2 + (x_2 - a_2)^2$ is a polynomial of degree 2, that is, $\deg(\{a\}) = 2$.

2. A union of two different lines.

Let $L, K \subseteq \mathbb{R}^2$ be two lines and let $x = (x_1, x_2) \in \mathbb{R}^2$. Take

$$L : \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 = 0, \quad K : \beta_0 + \beta_1 x_1 + \beta_2 x_2 = 0.$$

Then

$$\begin{aligned} x \in L \cup K \ &\Leftrightarrow \ \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 = 0 \ \vee \ \beta_0 + \beta_1 x_1 + \beta_2 x_2 = 0 \\ &\Leftrightarrow \ (\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2)(\beta_0 + \beta_1 x_1 + \beta_2 x_2) = 0. \end{aligned}$$

So

$$L \cup K : (\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2)(\beta_0 + \beta_1 x_1 + \beta_2 x_2) = 0$$

and $\varphi(x) = (\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2)(\beta_0 + \beta_1 x_1 + \beta_2 x_2)$ is a polynomial of degree 2, that is, $\deg(L \cup K) = 2$.

3. A circle.

Let $a = (a_1, a_2) \in \mathbb{R}^2$, $r > 0$ and $x = (x_1, x_2) \in \mathbb{R}^2$. A circle is defined in the following way:

$$S = S(a, r) \underset{df}{=} \{x \in \mathbb{R}^2 : \rho(x, a) = r\}.$$

Then a is called a *centre* of S and r is called a *radius* of S. Hence

$$x \in S \;\Leftrightarrow\; \rho(x, a) = r \;\Leftrightarrow\; [\rho(x, a)]^2 = r^2$$
$$\Leftrightarrow\; (x_1 - a_1)^2 + (x_2 - a_2)^2 = r^2.$$

So

$$S : (x_1 - a_1)^2 + (x_2 - a_2)^2 - r^2 = 0$$

and $\varphi(x) = (x_1 - a_1)^2 + (x_2 - a_2)^2 - r^2$ is a polynomial of degree 2, that is, $\deg(S) = 2$.

4. A conic.

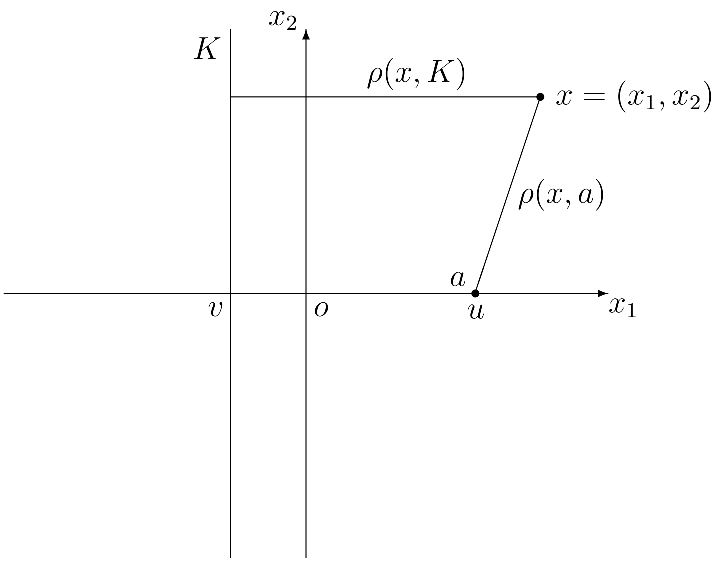
Definition. (**Conic**)

Let $a \in \mathbb{R}^2$, $K \subseteq \mathbb{R}^2$ be a line, $a \notin K$ and $e > 0$. The set

$$S(a, K, e) \underset{df}{=} \{x \in \mathbb{R}^2 : \rho(x, a) = e \cdot \rho(x, K)\}$$

is called the *conic*. Then, a is called a *focus* of $S(a, K, e)$, K is called a *directrix* of $S(a, K, e)$ and e is called an *eccentric* of $S(a, K, e)$.

Let us take such a coordinate system that the x_1-axis passes through the focus a and it is perpendicular to the directrix K, that is, $a = (u, 0)$, $K : x_1 - v = 0$ and $|u - v| = d$:



Then

$$\rho(x,a) = e\cdot\rho(x,K) \;\Leftrightarrow\; [\rho(x,a)]^2 = e^2\cdot[\rho(x,K)]^2,$$

that is,

$$(x_1-u)^2 + x_2^2 = e^2(x_1-v)^2.$$

Hence

$$S(a,K,e) : (1-e^2)x_1^2 + x_2^2 + 2(e^2v-u)x_1 + (u^2-e^2v^2) = 0$$

and $\varphi(x) = (1-e^2)x_1^2 + x_2^2 + 2(e^2v-u)x_1 + (u^2-e^2v^2)$ is a polynomial of degree 2, that is, $\deg(S(a,K,e)) = 2$.

Theorem. A conic, its focus, directrix and eccentric are invariants of isometry.

Proof. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an isometry. We have a conic:

$$S(a,K,e) = \{x\in\mathbb{R}^2 : \rho(x,a) = e\cdot\rho(x,K)\},$$

where a is its focus, K is its directrix and e is its eccentric. Then $f(K)$ is a line and

$$\rho(f(x),f(a)) = \rho(x,a) = e\cdot\rho(x,K) = e\cdot\rho(f(x),f(K)).$$

Hence

$$f(S(a,K,e)) = S(f(a),f(K),e) = \{y=f(x)\in\mathbb{R}^2 : \rho(y,f(a)) = e\cdot\rho(y,f(K))\}$$

is a conic which has a focus $f(a)$, a directrix $f(K)$ and an eccentric e. $\square$

Exercise. Show that a conic, its focus, directrix and eccentric are similarity invariants.

Definition.

A conic $S(a,K,e)$ is : 1) an *ellipse* if $e<1$,
2) a *parabola* if $e=1$,
3) a *hyperbola* if $e>1$.

We know that $a=(u,0)$, $K : x_1 - v = 0$, $|u-v| = d$ and

$$S(a,K,e) : (1-e^2)x_1^2 + x_2^2 + 2(e^2v-u)x_1 + (u^2-e^2v^2) = 0.$$

Parabola P:

Take $e=1$ and let $u=\frac{1}{2}d$ and $v=-\frac{1}{2}d$. Then

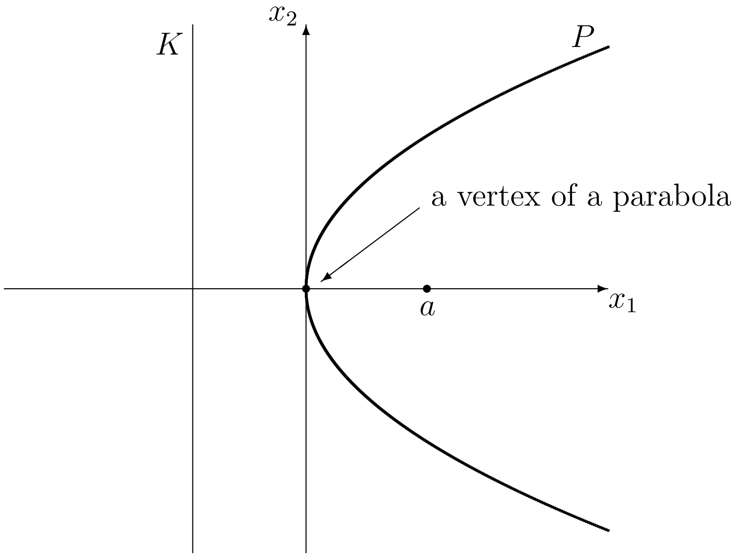
$$P : x_2^2 + 2(v-u)x_1 + (u^2-v^2) = 0,$$

that is,

$$P : x_2^2 - 2dx_1 = 0.$$

That is the *canonical equation of a parabola.*

It is easy to see that a parabola has one axis of symmetry: in canonical position the x_1-axis; does not have centres of symmetry; has a vertex, so a point of intersection of a parabola and its axis of symmetry: in canonical position point $(0,0)$; has one focus: in canonical position $a = (\frac{d}{2}, 0)$ and has one directrix: in canonical position $K : x_1 + \frac{d}{2} = 0$.



Ellipse E:

Take $e < 1$ and let $v - u = d$ and $u - e^2 v = 0$. Hence

$$u = \frac{e^2 d}{1-e^2}, \quad v = \frac{d}{1-e^2} \quad \text{and} \quad u, v > 0.$$

Then

$$u^2 - e^2 v^2 = \frac{(e^2 d)^2}{(1-e^2)^2} - \frac{e^2 d^2}{(1-e^2)^2} = -ud.$$

Thus

$$E : \frac{(1-e^2)x_1^2}{ud} + \frac{x_2^2}{ud} = 1.$$

Set: $\alpha_1 = \sqrt{\frac{ud}{1-e^2}}$ and $\alpha_2 = \sqrt{ud}$, where

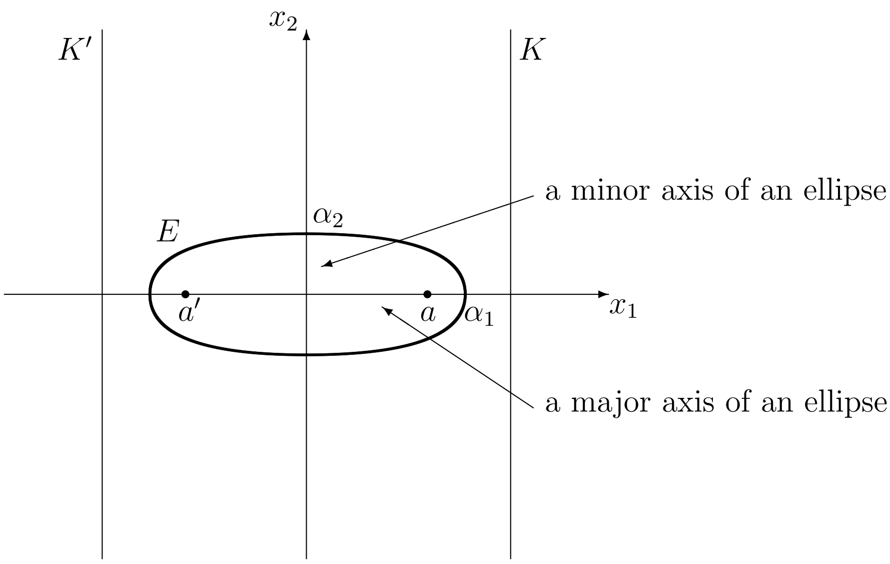
$$\alpha_1 = \frac{ed}{1-e^2} > 0, \quad \alpha_2 = \frac{ed}{\sqrt{1-e^2}} = \alpha_1 \sqrt{1-e^2} < \alpha_1.$$

Then

$$E : \frac{x_1^2}{\alpha_1^2} + \frac{x_2^2}{\alpha_2^2} = 1.$$

That is the *canonical equation of an ellipse.*

It is easy to see that an ellipse has two axes of symmetry: in canonical position the coordinate axes; has one centre of symmetry: in canonical position point $(0,0)$; has two foci: in canonical position $a = (\sqrt{\alpha_1^2 - \alpha_2^2}, 0)$ and $a' = (-\sqrt{\alpha_1^2 - \alpha_2^2}, 0)$ and has two directrices: in canonical position $K : x_1 - \frac{\alpha_1^2}{\sqrt{\alpha_1^2-\alpha_2^2}} = 0$ and $K' : x_1 + \frac{\alpha_1^2}{\sqrt{\alpha_1^2-\alpha_2^2}} = 0$. Moreover the eccentric $e = \frac{\sqrt{\alpha_1^2-\alpha_2^2}}{\alpha_1}$.



Remark. A circle is an ellipse (with $\alpha_1 = \alpha_2$).

Hyperbola H:

Take $e > 1$ and let $v - u = d$ and $u - e^2 v = 0$. Hence

$$u = \frac{e^2 d}{1 - e^2}, \quad v = \frac{d}{1 - e^2} \quad \text{and} \quad u, v < 0.$$

Then

$$u^2 - e^2 v^2 = -ud.$$

Thus

$$H : \frac{(1 - e^2)x_1^2}{ud} + \frac{x_2^2}{ud} = 1.$$

Setting $\alpha_1 = \sqrt{\frac{ud}{1-e^2}}$ and $\alpha_2 = \sqrt{-ud}$, where

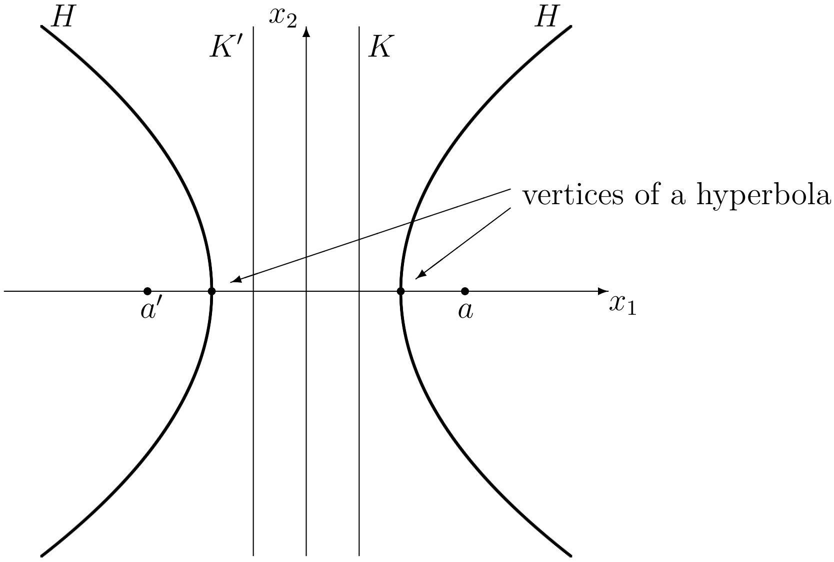
$$\alpha_1 = \frac{ed}{e^2 - 1} < -u, \quad \alpha_2 = \frac{ed}{\sqrt{e^2 - 1}} = \alpha_1 \sqrt{e^2 - 1} > \alpha_1,$$

we get

$$H : \frac{x_1^2}{\alpha_1^2} - \frac{x_2^2}{\alpha_2^2} = 1.$$

That is the *canonical equation of a hyperbola.*

It is easy to see that a hyperbola has two axes of symmetry: in canonical position the coordinate axes; has one centre of symmetry: in canonical position point $(0,0)$; has two foci: in canonical position $a = (\sqrt{\alpha_1^2 + \alpha_2^2}, 0)$ and $a' = (-\sqrt{\alpha_1^2 + \alpha_2^2}, 0)$ and has two directrices: in canonical position $K : x_1 - \frac{\alpha_1^2}{\sqrt{\alpha_1^2+\alpha_2^2}} = 0$ and $K' : x_1 + \frac{\alpha_1^2}{\sqrt{\alpha_1^2+\alpha_2^2}} = 0$. Moreover the eccentric $e = \frac{\sqrt{\alpha_1^2+\alpha_2^2}}{\alpha_1}$.



Definition. Let $F, F' \subseteq \mathbb{R}^n$ be algebraic sets of degree k. Then

F, F' are *isometric* $\underset{df}{\Leftrightarrow}$ there is an isometry $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(F) = F'$.

F, F' are *similar* $\underset{df}{\Leftrightarrow}$ there is a similarity $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(F) = F'$.

F, F' are identical from the *affine* point of view $\underset{df}{\Leftrightarrow}$ there is an affine transformation $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(F) = F'$.

Remark. Isometric sets are similar, and similar sets are identical from the affine point of view.

Theorem. All parabolas are similar.

Proof. Take a similarity $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$f(x) = \lambda x, \text{ where } \lambda > 0,$$

that is,

$$(\overline{x}_1, \overline{x}_2) = f(x_1, x_2) = (\lambda x_1, \lambda x_2).$$

Take a parabola $P : x_2^2 - 2dx_1 = 0$. Then

$$(\lambda x_2)^2 - 2\lambda d \cdot (\lambda x_1) = 0.$$

Hence $P' : \overline{x}_2^2 - 2\lambda d\overline{x}_1 = 0$ and $\lambda d = d' \;\Rightarrow\; \lambda = \frac{d'}{d}$.

Thus the similarity f transforms the parabola P onto the parabola P'. $\square$

Theorem. All ellipses are identical from the affine point of view.

Proof. Take an affine transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$(\overline{x}_1, \overline{x}_2) = f(x_1, x_2) = (x_1, \sqrt{1-e^2}\; x_2), \quad 0 < e < 1.$$

It is seen that f transforms the circle $S(0, \alpha_1) : x_1^2 + x_2^2 = \alpha_1^2$ onto the ellipse $E : \overline{x}_1^2 + \frac{\overline{x}_2^2}{1-e^2} = \alpha_1^2$, that is, onto the ellipse $E : \frac{\overline{x}_1^2}{\alpha_1^2} + \frac{\overline{x}_2^2}{\alpha_2^2} = 1$ (since $\alpha_2 = \alpha_1\sqrt{1-e^2}$). Hence every ellipse is an affine image of the circle. Thus all ellipses are identical from the affine point of view. $\square$

Theorem. All hyperbolas are identical from the affine point of view.

Proof. Take an affine transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$(\overline{x}_1, \overline{x}_2) = f(x_1, x_2) = (\alpha_1 x_1, \alpha_2 x_2).$$

It is seen that f transforms the hyperbola $H_0 : x_1^2 - x_2^2 = 1$ onto the hyperbola $H : \frac{\overline{x}_1^2}{\alpha_1^2} - \frac{\overline{x}_2^2}{\alpha_2^2} = 1$. Hence every hyperbola is an affine image of the hyperbola H_0. Thus all hyperbolas are the same from the affine point of view. $\square$

Finally we give a classification of algebraic sets of degree 2 in $\mathbb{R}^2$. Let us take a general equation of an algebraic set of degree 2 in $\mathbb{R}^2$:

$$\alpha_{11}x_1^2 + 2\alpha_{12}x_1x_2 + \alpha_{22}x_2^2 + 2\alpha_{13}x_1 + 2\alpha_{23}x_2 + \alpha_{33} = 0,$$

where $\alpha_{11}^2 + \alpha_{12}^2 + \alpha_{22}^2 > 0$.
Set:

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}, \quad \text{where} \quad \alpha_{21} = \alpha_{12}$$

and

$$\widetilde{A} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix}, \quad \text{where} \quad \alpha_{21} = \alpha_{12},\ \alpha_{31} = \alpha_{13},\ \alpha_{32} = \alpha_{23}.$$

Let $\det(A) = \Delta$, $\det\left(\widetilde{A}\right) = \widetilde{\Delta}$, $r(A) = k$ and $r\left(\widetilde{A}\right) = l$. Obviously, $0 \le k \le l$. Moreover, let

$$A_{11} = \begin{vmatrix} \alpha_{22} & \alpha_{23} \\ \alpha_{32} & \alpha_{33} \end{vmatrix} \quad \text{and} \quad A_{22} = \begin{vmatrix} \alpha_{11} & \alpha_{13} \\ \alpha_{31} & \alpha_{33} \end{vmatrix}.$$

Classification of algebraic sets of degree 2 in $\mathbb{R}^2$:

$[k,l]$			
$[2,3]$	$\Delta > 0,\ \alpha_{11}\widetilde{\Delta} < 0$	$\frac{x_1^2}{\alpha_1^2} + \frac{x_2^2}{\alpha_2^2} = 1$	ellipse
	$\Delta > 0,\ \alpha_{11}\widetilde{\Delta} > 0$	$\frac{x_1^2}{\alpha_1^2} + \frac{x_2^2}{\alpha_2^2} = -1$	empty set
	$\Delta < 0$	$\frac{x_1^2}{\alpha_1^2} - \frac{x_2^2}{\alpha_2^2} = 1$	hyperbola
$[2,2]$	$\Delta < 0$	$\frac{x_1^2}{\alpha_1^2} - \frac{x_2^2}{\alpha_2^2} = 0$	pair of intersecting lines
	$\Delta > 0$	$\frac{x_1^2}{\alpha_1^2} + \frac{x_2^2}{\alpha_2^2} = 0$	point
$[1,3]$	$\Delta = 0,\ \widetilde{\Delta} \neq 0$	$x_2^2 - 2dx_1 = 0$	parabola
$[1,2]$	$A_{22} < 0$ or $A_{11} < 0$	$x_2^2 - \alpha_2^2 = 0$	pair of parallel lines
	$A_{22} > 0,\ A_{11} > 0$	$x_2^2 + \alpha_2^2 = 0$	empty set
$[1,1]$		$x_2^2 = 0$	(double) line

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