Linear algebra with geometry II

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Preliminaries

The teaching script was created from lectures of the course Linear algebra with geometry II, which author have on KUL. That course is a continuation of the course Linear algebra with geometry I, whose teaching script is planned to write by author. There is a lot of material, because that course covers 60 hours. First there are two topics of linear algebra and next geometry with classification of algebraic sets of degree ≤ 2 in complex projective space. Discussed notions are given in understanding form and often illustrated by examples. Author hopes that teaching script will by helpfull for student.

1. Inner product spaces

Definition. (**Normed vector space**) $\mathbb{F} = \mathbb{R}$ (or $= \mathbb{C}$), V − a vector space over $\mathbb{F}$

A *normed vector space* is a vector space V equipped with a norm. A *norm* is a function $\|\cdot\| : V \to \mathbb{R}$, which satisfies the following properties, for all $v, w \in V$ and $\alpha \in \mathbb{F}$:

1) $\|\alpha v\| = |\alpha| \, \|v\|$. (Homogeneity).
2) $\|v\| \geq 0$ and $\|v\| = 0 \Leftrightarrow v = 0$. (Positive definiteness).
3) $\|v + w\| \leq \|v\| + \|w\|$. (Triangle inequality).

Example. Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. On the vector space $\mathbb{R}^n$ we define the following norms.

The 2-norm:

$$\|x\|_2 \underset{df}{=} \sqrt{x_1^2 + \ldots + x_n^2}.$$

So, for $n = 1$, we have $\|x\|_2 = |x|$.

The 1-norm:

$$\|x\|_1 \underset{df}{=} \sum_{i=1}^{n} |x_i| \, .$$

The ∞-norm:

$$\|x\|_\infty \underset{df}{=} \max_{i=1,\ldots,n} |x_i| \, .$$

Definition. (**Inner product space**) $\mathbb{F} = \mathbb{R}$ (or $= \mathbb{C}$), V − a vector space over $\mathbb{F}$

An *inner product space* is a vector space V equipped with an inner product (a scalar product). An *inner product* is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$, which satisfies the following properties, for all $v, v', w \in V$ and $\alpha \in \mathbb{F}$:

1) $\langle v + v', w \rangle = \langle v, w \rangle + \langle v', w \rangle$. (Linearity in the first argument).
2) $\langle \alpha v, w \rangle = \alpha \langle v, w \rangle$. (Homogeneity in the first argument).
3) $\langle v, v \rangle \geq 0$. (Positivity).
4) $\langle v, w \rangle = \overline{\langle w, v \rangle}$. (Conjugate symmetry).

Proposition. $v, v', w \in V$, $\alpha \in \mathbb{F}$

The following hold:

5) $\langle w, v + v' \rangle = \langle w, v \rangle + \langle w, v' \rangle$. (Linearity in the second argument).
6) $\langle v, \alpha w \rangle = \overline{\alpha} \langle v, w \rangle$.
7) $\langle v, 0 \rangle = \langle 0, v \rangle = 0$.

8) $\langle v, v\rangle = 0 \Leftrightarrow v = 0$.

Proof. 5) $\langle w, v+v'\rangle \overset{4)}{=} \overline{\langle v+v', w\rangle} \overset{1)}{=} \overline{\langle v, w\rangle + \langle v', w\rangle} = \overline{\langle v, w\rangle} + \overline{\langle v', w\rangle} \overset{4)}{=} \langle w, v\rangle + \langle w, v'\rangle$.

6) $\langle v, \alpha w\rangle \overset{4)}{=} \overline{\langle \alpha w, v\rangle} \overset{2)}{=} \overline{\alpha \langle w, v\rangle} = \overline{\alpha} \cdot \overline{\langle w, v\rangle} \overset{4)}{=} \overline{\alpha} \cdot \langle v, w\rangle$.

7) $\langle v, 0\rangle = \langle v, 0+0\rangle \overset{5)}{=} \langle v, 0\rangle + \langle v, 0\rangle \;\Rightarrow\; \langle v, 0\rangle = 0$.
Similarly, $\langle 0, v\rangle = 0$.

8) ($\Rightarrow$) If $\langle v, v\rangle = 0$, then, by 7), $\langle v, v\rangle = \langle v, 0\rangle = \langle 0, v\rangle$, whence $v = 0$.

($\Leftarrow$) Follows by 7). $\square$

Remark. If $\mathbb{F} = \mathbb{R}$, then property 4) says that $\langle v, w\rangle = \langle w, v\rangle$. An inner product space $V(\mathbb{F})$ such that $\dim V < \infty$, is called *Euclidean space* if $\mathbb{F} = \mathbb{R}$, and *unitary space* if $\mathbb{F} = \mathbb{C}$.

Example. Let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. On the vector space $\mathbb{R}^n$ we define the standard inner product by

$$\langle x, y\rangle \underset{df}{=} \sum_{i=1}^{n} x_i y_i.$$

More generally, if $\alpha_1, \ldots, \alpha_n > 0$, then the following definition also gives an inner product

$$\langle x, y\rangle_\alpha \underset{df}{=} \sum_{i=1}^{n} \alpha_i x_i y_i.$$

Example. Let $w = (w_1, \ldots, w_n), z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. On the vector space $\mathbb{C}^n$ we define the standard inner product by

$$\langle w, z\rangle \underset{df}{=} \sum_{i=1}^{n} w_i \overline{z_i}.$$

Example. Let $A, B \in M_{n\times n}(\mathbb{R})$. On the vector space $M_{n\times n}(\mathbb{R})$ we define the standard inner product by

$$\langle A, B\rangle \underset{df}{=} \operatorname{tr}\left(B^T A\right).$$

Definition. (**Orthogonal vectors**) V – an inner product space, $v, w \in V$

v, w are *orthogonal* (or *perpendicular*) $\underset{df}{\Leftrightarrow}$ $\langle v, w\rangle = 0$.

We write $v \perp w$.

Proposition. (**Cauchy-Schwarz inequality**) V – an inner product space, $v, w \in V$

Then

$$|\langle v, w\rangle| \leq \sqrt{\langle v, v\rangle}\sqrt{\langle w, w\rangle}.$$

Proof. If $w = 0$, then

$$|\langle v, 0\rangle| = 0 \leq \sqrt{\langle v, v\rangle}\sqrt{\langle 0, 0\rangle} = 0.$$

Assume that $w \neq 0$. So, $\langle w, w\rangle > 0$. Define

$$\alpha = \frac{\langle v, w\rangle}{\langle w, w\rangle}.$$

We have

$$\langle v, w\rangle \langle w, v\rangle = \langle v, w\rangle \overline{\langle v, w\rangle} = |\langle v, w\rangle|^2$$

and

$$\langle v - \alpha w, v - \alpha w\rangle \geq 0,$$

that is,

$$\langle v, v\rangle - \alpha \langle w, v\rangle - \overline{\alpha} \langle v, w\rangle + |\alpha|^2 \langle w, w\rangle \geq 0.$$

Hence,

$$\langle v, v\rangle - \frac{|\langle v, w\rangle|^2}{\langle w, w\rangle} - \frac{|\langle v, w\rangle|^2}{\langle w, w\rangle} + \frac{|\langle v, w\rangle|^2}{\langle w, w\rangle} \geq 0.$$

So,

$$\langle v, v\rangle - \frac{|\langle v, w\rangle|^2}{\langle w, w\rangle} \geq 0.$$

Thus,

$$|\langle v, w\rangle| \leq \sqrt{\langle v, v\rangle}\sqrt{\langle w, w\rangle}. \quad \square$$

Proposition. Let $\langle \cdot, \cdot\rangle$ be an inner product on a vector space V. Then the function $\|\cdot\| : V \to \mathbb{R}$ defined by $\|v\| = \sqrt{\langle v, v\rangle}$ is a norm on V, so an inner product space is a normed vector space.

Proof. We easily have homogeneity and positive definiteness. Let $v, w \in V$. We prove triangle inequality. We have

$$\begin{aligned}\|v+w\|^2 &= |\langle v+w, v+w\rangle| \\ &= |\langle v, v+w\rangle + \langle w, v+w\rangle| \\ &= |\langle v,v\rangle + \langle w,w\rangle + \langle v,w\rangle + \langle w,v\rangle| \\ &\le |\langle v,v\rangle| + |\langle w,w\rangle| + |\langle v,w\rangle| + |\langle w,v\rangle| \\ &\le \|v\|^2 + \|w\|^2 + 2\sqrt{\langle v,v\rangle}\cdot\sqrt{\langle w,w\rangle} \\ &= \|v\|^2 + \|w\|^2 + 2\|v\|\cdot\|w\| \\ &= (\|v\| + \|w\|)^2 .\end{aligned}$$

Hence,

$$\|v+w\| \le \|v\| + \|w\| .$$

Thus an inner product space is a normed vector space. □

Proposition. V – an inner product space, $v, v_1, \ldots, v_n \in V$

If $v \perp v_i$ for any $i = 1, \ldots, n$, then

$$v \perp \sum_{i=1}^{n} \alpha_i v_i \quad \text{for any} \quad \alpha_1, \ldots, \alpha_n \in \mathbb{F}.$$

Proof. Let $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$. Since $\langle v, v_i\rangle = 0$ for any $i = 1, \ldots, n$, we have

$$\begin{aligned}\left\langle v, \sum_{i=1}^{n} \alpha_i v_i \right\rangle &= \langle v, \alpha_1 v_1 + \ldots + \alpha_n v_n\rangle \\ &= \langle v, \alpha_1 v_1\rangle + \ldots + \langle v, \alpha_n v_n\rangle \\ &= \overline{\alpha_1}\langle v, v_1\rangle + \ldots + \overline{\alpha_n}\langle v, v_n\rangle \\ &= \sum_{i=1}^{n} \overline{\alpha_i}\langle v, v_i\rangle \\ &= 0,\end{aligned}$$

that is,

$$v \perp \sum_{i=1}^{n} \alpha_i v_i. \quad \square$$

Theorem. (**Pythagorean theorem**) V – an inner product space, $v, w \in V$

If $v \perp w$, then

$$\|v+w\|^2 = \|v\|^2 + \|w\|^2 .$$

Proof. Since $\langle v, w\rangle = 0$, we have

$$\|v+w\|^2 = \langle v+w, v+w\rangle = \langle v, v\rangle + \langle w, w\rangle + \langle v, w\rangle + \langle w, v\rangle = \|v\|^2 + \|w\|^2 . \quad \square$$

Theorem. (**Generalized Pythagorean theorem**) V – an inner product space, $v_1, \ldots, v_n \in V$

If $v_1, \ldots, v_n$ are orthogonal to each other, that is, $\langle v_i, v_j\rangle = 0$ for all $i, j = 1, \ldots, n$ and $i \neq j$, then

$$\left\| \sum_{i=1}^{n} v_i \right\|^2 = \sum_{i=1}^{n} \|v_i\|^2 .$$

Proof. We induct on n. For $n = 1$ it is easy. For $n = 2$ it is Pythagorean theorem. Suppose the assertion is true for a fixed n. Let $v_1, \ldots, v_{n+1} \in V$ be orthogonal to each other. We have

$$\left\langle v_{n+1}, \sum_{i=1}^{n} \alpha_i v_i \right\rangle = 0$$

and from Pythagorean theorem,

$$\begin{aligned} \left\| \sum_{i=1}^{n+1} v_i \right\|^2 &= \left\| v_{n+1} + \sum_{i=1}^{n} v_i \right\|^2 \\ &= \|v_{n+1}\|^2 + \left\| \sum_{i=1}^{n} v_i \right\|^2 \\ &= \|v_{n+1}\|^2 + \sum_{i=1}^{n} \|v_i\|^2 \\ &= \sum_{i=1}^{n+1} \|v_i\|^2 . \quad \square \end{aligned}$$

Conclusion. V – an inner product space, $v_1, \ldots, v_n \in V$, $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$

If $v_1, \ldots, v_n$ are orthogonal to each other, that is, $\langle v_i, v_j\rangle = 0$ for all $i, j = 1, \ldots, n$ and $i \neq j$, then

$$\left\| \sum_{i=1}^{n} \alpha_i v_i \right\|^2 = \sum_{i=1}^{n} |\alpha_i|^2 \|v_i\|^2 .$$

Conclusion. V – an inner product space, $v_1, \ldots, v_n \in V$

If $v_1, \ldots, v_n$ are orthogonal to each other, that is, $\langle v_i, v_j\rangle = 0$ for all $i, j = 1, \ldots, n$ and $i \neq j$, then the set $\{v_1, \ldots, v_n\}$ is linearly independent.

Definition. (**Orthogonal set, orthonormal set**) V – an inner product space, $v_1, \dots, v_n \in V$

The set $(v_1, \dots, v_n)$ is said to be *orthogonal* if $\langle v_i, v_j \rangle = 0$ for all $i, j = 1, \dots, n$ and $i \neq j$. An orthogonal set $(v_1, \dots, v_n)$ is said to be *orthonormal* if $\|v_i\| = 1$ for all $i = 1, \dots, n$.

Conclusion. V – an inner product space, $v_1, \dots, v_n \in V$, $(v_1, \dots, v_n)$ – orthonormal set

Then,

$$\left\| \sum_{i=1}^{n} \alpha_i v_i \right\|^2 = \sum_{i=1}^{n} |\alpha_i|^2 .$$

Conclusion. Any orthogonal (orthonormal) set is linearly independent.

Definition. (**Orthogonal basis, orthonormal basis**) V – an inner product space

Orthogonal basis of $V \underset{df}{=}$ a basis of V that is also an orthogonal set.
Orthonormal basis of $V \underset{df}{=}$ a basis of V that is also an orthonormal set.

Conclusion. V – an n-dimensional inner product space, $v_1, \dots, v_n \in V$,
$(v_1, \dots, v_n)$ – orthogonal (orthonormal) set

Then $(v_1, \dots, v_n)$ is an orthogonal (orthonormal) basis of V.

Remark. If $(v_1, \dots, v_n)$ is an orthogonal basis of an inner product space V, then $\left(\frac{v_1}{\|v_1\|}, \dots, \frac{v_n}{\|v_n\|} \right)$ is an orthonormal basis of V, because $\left\| \frac{v_i}{\|v_i\|} \right\| = \frac{\|v_i\|}{\|v_i\|} = 1$ for all $i = 1, \dots, n$.

Theorem. V – an inner product space, $(v_1, \dots, v_n)$ – an orthogonal basis of V

Then, for any $v \in V$,

$$v = \sum_{i=1}^{n} \frac{\langle v, v_i \rangle}{\|v_i\|^2} v_i.$$

Proof. Let $v \in V$ and $\left(\frac{v_1}{\|v_1\|}, \dots, \frac{v_n}{\|v_n\|} \right)$ be an orthonormal basis of V. Then there exist $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ such that

$$v = \sum_{i=1}^{n} \alpha_i \frac{v_i}{\|v_i\|}.$$

We show that $\alpha_i = \frac{\langle v, v_i \rangle}{\|v_i\|}$ for all $i = 1, \dots, n$. For any $j = 1, \dots, n$ we have

$$\begin{aligned} \langle v, v_j \rangle &= \left\langle \sum_{i=1}^{n} \alpha_i \frac{v_i}{\|v_i\|}, v_j \right\rangle = \sum_{i=1}^{n} \frac{\alpha_i}{\|v_i\|} \langle v_i, v_j \rangle \\ &= \frac{\alpha_j}{\|v_j\|} \langle v_j, v_j \rangle = \frac{\alpha_j}{\|v_j\|} \|v_j\|^2 = \alpha_j \|v_j\| , \end{aligned}$$

whence $\alpha_j = \frac{\langle v, v_j \rangle}{\|v_j\|}$ for all $j = 1, \ldots, n$. $\square$

Conclusion. V – an inner product space, $(v_1, \ldots, v_n)$ – an orthonormal basis of V

Then, for any $v \in V$,

$$v = \sum_{i=1}^{n} \langle v, v_i \rangle v_i.$$

Conclusion. V – an inner product space, $\mathcal{B} = (v_1, \ldots, v_n)$ – an orthonormal basis of V

Then, for any $v \in V$,

$$[v]^{\mathcal{B}} = \begin{bmatrix} \langle v, v_1 \rangle \\ \vdots \\ \langle v, v_n \rangle \end{bmatrix}.$$

Definition. (**Unit vector**) V – a normed vector space, $v \in V$

v is a *unit vector* $\underset{df}{\Leftrightarrow}$ $\|v\| = 1$.

Remark. Let $v \neq 0$. Then $\frac{v}{\|v\|}$ is a unit vector.

Definition. (**Projection onto a vector**) V – an inner product space, $v, w \in V$, $w \neq 0$

Define the *orthogonal projection* of v onto w by

$$\mathrm{P}_w(v) \underset{df}{=} \frac{\langle v, w \rangle}{\|w\|^2} w.$$

Note that P_w is a linear transformation.

Definition. (**Projection onto a subspace**) V – an inner product space, $v \in V$, $W \subseteq V$ – an n-dimensional subspace of V, $(w_1, \ldots, w_n)$ – an orthogonal basis of W

Define the *orthogonal projection* of v onto W by

$$\mathrm{P}_W(v) \underset{df}{=} \sum_{i=1}^{n} \frac{\langle v, w_i \rangle}{\|w_i\|^2} w_i.$$

Note that $\mathrm{P}_W : V \to V$ is a linear transformation, and $R(\mathrm{P}_W) \subseteq W$.

Remark. $\mathrm{P}_W(v) = v$ iff $v \in W$. Also, the definition of $\mathrm{P}_W(v)$ does not depend on the orthogonal basis $(w_1, \ldots, w_n)$.

Remark. V – an inner product space, $v \in V$, $W \subseteq V$ – an n-dimensional subspace of V, $(w_1, \ldots, w_n)$ – an orthogonal set of nonzero vectors in W

We can write

$$v = (v - \mathrm{P}_W(v)) + \mathrm{P}_W(v).$$

Note that $\mathrm{P}_W(v) \in W$ and $(v - \mathrm{P}_W(v))$ is orthogonal to w_i for each $i = 1, \ldots, n$. So, $(v - \mathrm{P}_W(v))$ is orthogonal to any vector in W.

Conclusion. V – an inner product space, $v \in V$, $W \subseteq V$ – an n-dimensional subspace of V, $(w_1, \ldots, w_n)$ – an orthonormal basis of W

Then

$$\mathrm{P}_W(v) \underset{df}{=} \sum_{i=1}^{n} \langle v, w_i \rangle w_i.$$

Theorem. (**Gram-Schmidt Orthogonalization**) V – an inner product space, $\{v_1, \ldots, v_n\}$ – a linearly independent set

Let

$$\begin{aligned}
w_1 &= v_1 \\
w_2 &= v_2 - \mathrm{P}_{w_1}(v_2) = v_2 - \frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} w_1 \\
w_3 &= v_3 - \mathrm{P}_{\mathrm{span}(w_1, w_2)}(v_3) = v_3 - \left[\frac{\langle v_3, w_1 \rangle}{\|w_1\|^2} w_1 + \frac{\langle v_3, w_2 \rangle}{\|w_2\|^2} w_2 \right] \\
&\vdots \\
w_n &= v_n - \mathrm{P}_{\mathrm{span}(w_1, \ldots, w_{n-1})}(v_n) = v_n - \left[\frac{\langle v_n, w_1 \rangle}{\|w_1\|^2} w_1 + \cdots + \frac{\langle v_n, w_{n-1} \rangle}{\|w_{n-1}\|^2} w_{n-1} \right].
\end{aligned}$$

Then $\{w_1, \ldots, w_n\}$ is an orthogonal set of nonzero vectors in V. Also, $\mathrm{span}(w_1, \ldots, w_k) = \mathrm{span}(v_1, \ldots, v_k)$ for each $k = 1, \ldots, n$. Finally, note that the set $\left(\frac{w_1}{\|w_1\|}, \ldots, \frac{w_n}{\|w_n\|} \right)$ is an orthonormal set of vectors in V with the same span as $v_1, \ldots, v_n$.

Proof. We prove it by induction on $k = 2, \ldots, n$. We know that $w_2 \perp w_1$ by previous Remark. Assume that $\{w_1, \ldots, w_k\}$ is orthogonal, $w_1, \ldots, w_k$ are nonzero and $\mathrm{span}(w_1, \ldots, w_k) = \mathrm{span}(v_1, \ldots, v_k)$ for some k. Then

$$w_{k+1} = v_{k+1} - \mathrm{P}_{\mathrm{span}(w_1, \ldots, w_k)}(v_{k+1}) = v_{k+1} - \mathrm{P}_{\mathrm{span}(v_1, \ldots, v_k)}(v_{k+1}) \tag{1}$$

By previous Remark w_{k+1} is orthogonal to any vector in $\mathrm{span}(v_1, \ldots, v_k) = \mathrm{span}(w_1, \ldots, w_k)$. We have

$$v_{k+1} \notin \mathrm{span}(v_1, \ldots, v_k) \text{ (by linear independence)} \Rightarrow v_{k+1} \neq \mathrm{P}_{\mathrm{span}(v_1, \ldots, v_k)}(v_{k+1}) \Rightarrow w_{k+1} \neq 0.$$

Hence $\{w_1, \ldots, w_{k+1}\}$ is an orthogonal set of nonzero vectors. By (1), $w_{k+1} \in \operatorname{span}(v_1, \ldots, v_{k+1})$, that is,

$$\operatorname{span}(w_1, \ldots, w_{k+1}) \subseteq \operatorname{span}(v_1, \ldots, v_{k+1}).$$

Moreover, $\{w_1, \ldots, w_{k+1}\}$ and $\{v_1, \ldots, v_{k+1}\}$ are orthogonal sets, so bases. Hence

$$\operatorname{span}(w_1, \ldots, w_{k+1}) = \operatorname{span}(v_1, \ldots, v_{k+1}). \ \square$$

Conclusion. Every finite-dimensional inner product space has an orthogonal (orthonormal) basis.

Conclusion. V – an inner product space, $W \subseteq V$ – a finite-dimensional subspace of V

Then there exists a linear transformation $P : V \to V$ such that $P^2 = P$, $R(P) \subseteq W$ and $P(w) = w$ for any $w \in W$. That is, P is a projection onto W.

Definition. (**Orthogonal subspaces**) $V_1, V_2 \subseteq V$ – subspaces of an inner product space V

V_1 is orthogonal to V_2, $V_1 \perp V_2 \underset{df}{\Leftrightarrow} v_1 \perp v_2$ for all $v_1 \in V_1$ and $v_2 \in V_2$.

Proposition. $V_1, V_2 \subseteq V$ – subspaces of an inner product space V

Then,

$$V_1 \perp V_2 \Rightarrow V_1 \cap V_2 = \{0\}.$$

Proof. Since $0 \in V_1$ and $0 \in V_2$, it follows $0 \in V_1 \cap V_2$. Assume that $v \in V_1 \cap V_2$. Then, $v \in V_1$ and $v \in V_2$. Now, we know that if $v_1 \in V_1$ and $v_2 \in V_2$, then $v_1 \perp v_2$, that is, $\langle v_1, v_2 \rangle = 0$. In particular, $\langle v, v \rangle = 0$, that is, $v = 0$. Thus, $V_1 \cap V_2 = \{0\}$. $\square$

Definition. (**Orthogonal complement**) $V_1 \subseteq V$ – a subspace of an inner product space V

Define the *orthogonal complement* of V_1 in V by

$$V_1^\perp \underset{df}{=} \{v \in V : \langle v, v_1 \rangle = 0 \text{ for all } v_1 \in V_1\}.$$

Exercise. Show that $\{0\}^\perp = V$ and $V^\perp = \{0\}$.

Exercise. Let V_1 be a subspace of an inner product space V. Show that $V_1^\perp$ is a subspace of V.

The following Theorem gives an algorithm for computing orthogonal complements.

Theorem. V – an n-dimensional inner product space, $W \subseteq V$ – an k-dimensional subspace of V,
$(v_1, \ldots, v_k)$ – a basis of W, $(v_1, \ldots, v_n)$ – an extension of $(v_1, \ldots, v_k)$
$w_1, \ldots, w_n$ – orthonormal vectors produced by Gram-Schmidt Orthogonalization

Then $(w_1, \ldots, w_k)$ is an orthonormal basis of W and $(w_{k+1}, \ldots, w_n)$ is an orthonormal basis of $W^\perp$.

Proof. We know that $\operatorname{span}(w_1, \ldots, w_k) = \operatorname{span}(v_1, \ldots, v_k)$, so $(w_1, \ldots, w_k)$ is an orthonormal basis of W. We have $w_{k+1}, \ldots, w_n$ are orthonormal, so linearly independent.

$$W^\perp \stackrel{?}{=} \operatorname{span}(w_{k+1}, \ldots, w_n)$$

Let $j \in \{k+1, \ldots, n\}$. Then $w_j \perp w_i$ for all $i = 1, \ldots, k$ (by Gram-Schmidt Orthogonalization), that is, w_j is orthogonal to all vectors of W. Hence, $w_j \in W^\perp$. So, $\operatorname{span}(w_{k+1}, \ldots, w_n) \subseteq W^\perp$. Let $w \in W^\perp \subseteq V$. Hence,

$$w = \sum_{i=1}^{n} \langle w, w_i \rangle w_i.$$

Since $w \in W^\perp$, it follows $\langle w, w_i \rangle = 0$ for all $i = 1, \ldots, k$, that is,

$$w = \sum_{i=k+1}^{n} \langle w, w_i \rangle w_i.$$

So, $w \in \operatorname{span}(w_{k+1}, \ldots, w_n)$. Therefore,

$$W^\perp = \operatorname{span}(w_{k+1}, \ldots, w_n). \quad \Box$$

Conclusion. (**Dimension Theorem for orthogonal complements**)
V – a finite-dimensional inner product space, $W \subseteq V$ – a subspace of V

Then

$$\dim(W) + \dim(W^\perp) = \dim(V).$$

Conclusion. V – a finite-dimensional inner product space, $W \subseteq V$ – a subspace of V

Then every $v \in V$ can be written uniquely as $v = w_1 + w_2$, where $w_1 \in W$ and $w_2 \in W^\perp$.

2. Quadratic forms

Definition. $f : \mathbb{R}^n \to \mathbb{R}$ – a function

f is a *quadratic form* in $\mathbb{R}^n$ $\underset{df}{\Leftrightarrow}$

$$f(x) = \sum_{i,j=1}^{n} \alpha_{ij} x_i x_j$$

for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\alpha_{ij} \in \mathbb{R}$, $i, j = 1, \ldots, n$.

Examples.
1. $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x) = \alpha_{11}x_1^2 + \alpha_{12}x_1x_2 + \alpha_{22}x_2^2$, $x = (x_1, x_2)$.
2. $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x) = \alpha_{11}x_1^2 + \alpha_{22}x_2^2 + \alpha_{33}x_3^2 + \alpha_{12}x_1x_2 + \alpha_{13}x_1x_3 + \alpha_{23}x_2x_3$, $x = (x_1, x_2, x_3)$.

Theorem. $f : \mathbb{R}^n \to \mathbb{R}$ – a quadratic form

Then

1) $f(0) = 0$,
2) $f(\alpha x) = \alpha^2 f(x)$.

Proof. Easy. □

Theorem. $f : \mathbb{R}^n \to \mathbb{R}$ – a quadratic form

There exists a unique symmetric matrix $A = [\alpha_{ij}]_{n\times n}$ such that

$$f(x) = \sum_{i,j=1}^{n} \alpha_{ij} x_i x_j \quad \text{for} \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Moreover,

$$f(x) = \sum_{i=1}^{n} \alpha_{ii} x_i^2 + 2 \sum_{1\le i<j\le n}^{n} \alpha_{ij} x_i x_j.$$

Proof. Let $f(x) = \sum\limits_{i,j=1}^{n} \beta_{ij} x_i x_j$ and let $B = [\beta_{ij}]$. Then

$$f(x) = \sum_{i=1}^{n} \beta_{ii} x_i^2 + 2 \sum_{1\le i<j\le n}^{n} (\beta_{ij} + \beta_{ji}) x_i x_j = \sum_{i,j=1}^{n} \frac{\beta_{ij} + \beta_{ji}}{2} x_i x_j.$$

Thus, $A = \frac{1}{2}(B + B^T)$. Obviously, it is symmetric.

Now, we prove that A is unique. Let $C = [\gamma_{ij}]_{n\times n}$ be such that

$$f(x) = \sum_{i,j=1}^{n} \gamma_{ij} x_i x_j.$$

Then, $f(e_i) = \alpha_{ii} = \gamma_{ii}$ for $i = 1, \ldots, n$, where $(e_1, \ldots, e_n)$ is the standard basis of $\mathbb{R}^n$. Moreover,

$$f(e_i + e_j) = \alpha_{ii} + 2\alpha_{ij} + \alpha_{jj}$$

and

$$f(e_i + e_j) = \gamma_{ii} + 2\gamma_{ij} + \gamma_{jj} = \alpha_{ii} + 2\gamma_{ij} + \alpha_{jj},$$

that is, $\alpha_{ij} = \gamma_{ij}$ for all $i, j = 1, \ldots, n$.

Hence, $A = C$, that is, A is unique. $\square$

Definition. $f : \mathbb{R}^n \to \mathbb{R}$ – a quadratic form, $A = [\alpha_{ij}]_{n\times n}$ – a symmetric matrix

A is a *matrix of* f in the standard basis of $\mathbb{R}^n$ $\underset{df}{\Leftrightarrow}$

$$f(x) = \sum_{i,j=1}^{n} \alpha_{ij} x_i x_j = \sum_{i=1}^{n} \alpha_{ii} x_i^2 + 2 \sum_{1\le i<j\le n} \alpha_{ij} x_i x_j,$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

Remark. For a quadratic form $f : \mathbb{R}^n \to \mathbb{R}$ we can use the following matrix notation

$$f(x) = X^T \cdot A \cdot X,$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $X = [x_1 \cdots x_n]^T$.

Definition. $f : \mathbb{R}^n \to \mathbb{R}$ – a quadratic form, $\mathcal{B} = (v_1, \ldots, v_n)$ – a basis of $\mathbb{R}^n$

$B = [\beta_{ij}]_{n\times n}$ is a *matrix of* f in $\mathcal{B}$ $\underset{df}{\Leftrightarrow}$

$$f(x) = \sum_{i,j=1}^{n} \beta_{ij} y_i y_j = Y^T B Y,$$

where $x = y_1 v_1 + \ldots + y_n v_n$ and $Y = [y_1 \cdots y_n]^T$.

Then $f(x) = Y^T \cdot B \cdot Y$ is a matrix notation of f in a basis $\mathcal{B}$.

Theorem. $f:\mathbb{R}^n\to\mathbb{R}$ – a quadratic form, $\mathcal{A}$ – the standard basis of $\mathbb{R}^n$, $\mathcal{B}$ – some basis of $\mathbb{R}^n$

A – a matrix of f in $\mathcal{A}$, $Q=[I_{\mathbb{R}^n}]^{\mathcal{A}}_{\mathcal{B}}$

Then a matrix B of f in $\mathcal{B}$ has a form

$$B=Q^T\cdot A\cdot Q.$$

Proof. Let $x=(x_1,\ldots,x_n)\in\mathbb{R}^n$, $X=[x]^{\mathcal{A}}$, $Y=[x]^{\mathcal{B}}$. Then we know that $QY=X$. Hence,

$$f(x)=X^T\cdot A\cdot X=(QY)^T\cdot A\cdot(QY)=Y^T\cdot(Q^T\cdot A\cdot Q)\cdot Y$$

and

$$f(x)=Y^T\cdot B\cdot Y\ \text{ in a basis }\mathcal{B}.$$

Let $\mathcal{B}=(v_1,\ldots,v_n)$, $B=[\beta_{ij}]_{n\times n}$, $Q^T\cdot A\cdot Q=[\gamma_{ij}]_{n\times n}$. Then

$$f(v_i)=\gamma_{ii}=\beta_{ii}\ \text{ for }\ i=1,\ldots,n,$$

$$f(v_i+v_j)=\gamma_{ii}+2\gamma_{ij}+\gamma_{jj}$$

and

$$f(v_i+v_j)=\beta_{ii}+2\beta_{ij}+\beta_{jj}=\gamma_{ii}+2\beta_{ij}+\gamma_{jj}$$

for $i,j=1,\ldots,n$ such that $i\neq j$, that is, $\gamma_{ij}=\beta_{ij}$ for all $i,j=1,\ldots,n$.

Hence, $Q^TAQ=B$. $\square$

Definition. $f:\mathbb{R}^n\to\mathbb{R}$ – a quadratic form, $\mathcal{B}$ – a basis of $\mathbb{R}^n$

f has a *canonical form* in $\mathcal{B}$ $\underset{df}{\Leftrightarrow}$

$$\bigvee_{\delta_1,\ldots,\delta_n\in\mathbb{R}}\ \bigwedge_{x\in\mathbb{R}^n} f(x)=\delta_1y_1^2+\ldots+\delta_ny_n^2,$$

where $[x]^{\mathcal{B}}=[y_1\cdots y_n]^T$, $\delta_1,\ldots,\delta_n$ – coefficients.

Definition. $f:\mathbb{R}^n\to\mathbb{R}$ – a quadratic form

A *canonical basis* of f $\underset{df}{=}$ any basis of $\mathbb{R}^n$ in which f has a canonical form.

Remark. A quadratic form can have many canonical bases and many canonical forms. A matrix of a quadratic form in its canonical basis is diagonal.

Theorem. Any quadratic form in $\mathbb{R}^n$ has a canonical basis.

Remark. As proof of this theorem we present Lagrange's method of finding a canonical form of a quadratic form.

Lagrange's method:

Let $f:\mathbb{R}^n \to \mathbb{R}$ and

$$f(x)=\sum_{i=1}^{n}\alpha_{ii}x_i^2+2\sum_{1\le i<j\le n}^{n}\alpha_{ij}x_ix_j.$$

1. $\bigvee_i \alpha_{ii}\neq 0$

Assume that $\alpha_{11}\neq 0$. We group all terms with x_1:

$$\left(\alpha_{11}x_1^2+2\alpha_{12}x_1x_2+\ldots+2\alpha_{1n}x_1x_n\right)+h(x_2,\ldots,x_n)$$

and next we multiply and divide the expression in brackets by α_{11}:

$$\frac{1}{\alpha_{11}}\left(\alpha_{11}^2x_1^2+2\alpha_{11}\alpha_{12}x_1x_2+\ldots+2\alpha_{11}\alpha_{1n}x_1x_n\right)+h(x_2,\ldots,x_n).$$

Now, we use the following formula

$$(x_1+x_2+\ldots+x_n)^2=x_1^2+x_2^2+\ldots+x_n^2+2\sum_{1\le i<j\le n}^{n}x_ix_j$$

and get

$$\frac{1}{\alpha_{11}}(\alpha_{11}x_1+\alpha_{12}x_2+\ldots+\alpha_{1n}x_n)^2-\frac{1}{\alpha_{11}}\left(\alpha_{12}^2x_2^2+\ldots+\alpha_{1n}^2x_n^2+2\alpha_{12}\alpha_{13}x_2x_3+\ldots\right)+h(x_2,\ldots,x_n)$$
$$=\frac{1}{\alpha_{11}}(\alpha_{11}x_1+\alpha_{12}x_2+\ldots+\alpha_{1n}x_n)^2+h_1(x_2,\ldots,x_n),$$

where $h_1(x_2,\ldots,x_n)=-\frac{1}{\alpha_{11}}\left(\alpha_{12}^2x_2^2+\ldots+\alpha_{1n}^2x_n^2+2\alpha_{12}\alpha_{13}x_2x_3+\ldots\right)+h(x_2,\ldots,x_n)$ is a quadratic form in $\mathbb{R}^{n-1}$.

We continue and finally make substitution to obtain a canonical form of f.

2. $\bigwedge_i \alpha_{ii}=0$

Assume that $\alpha_{12}\neq 0$. We put

$$x_1=y_1+y_2,\ x_2=y_1-y_2,\ x_3=y_3,\ \ldots,\ x_n=y_n.$$

Then

$$2\alpha_{12}x_1x_2=2\alpha_{12}(y_1+y_2)(y_1-y_2)=2\alpha_{12}y_1^2-2\alpha_{12}y_2^2.$$

Further we make point **1.**

Example. Using the Lagrange's method find a canonical form of a quadratic form $f : \mathbb{R}^3 \to \mathbb{R}$ such that

$$f(x) = 2x_1^2 - x_2^2 + 3x_3^2 + 2x_1x_2 - 4x_1x_3 - 3x_2x_3 \quad \text{for} \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3.$$

Solution.

We have

$$\begin{aligned}
f(x) &= 2x_1^2 - x_2^2 + 3x_3^2 + 2x_1x_2 - 4x_1x_3 - 3x_2x_3 \\
&= 2\left(x_1^2 + x_1x_2 - 2x_1x_3\right) - x_2^2 + 3x_3^2 - 3x_2x_3 \\
&= 2\left(x_1^2 + \frac{1}{4}x_2^2 + x_3^2 + x_1x_2 - 2x_1x_3 - x_2x_3\right) - \frac{1}{2}x_2^2 - 2x_3^2 + 2x_2x_3 - x_2^2 + 3x_3^2 - 3x_2x_3 \\
&= 2\left(x_1 + \frac{1}{2}x_2 - x_3\right)^2 - \frac{3}{2}x_2^2 + x_3^2 - x_2x_3 \\
&= 2\left(x_1 + \frac{1}{2}x_2 - x_3\right)^2 - \frac{3}{2}\left(x_2^2 + \frac{2}{3}x_2x_3\right) + x_3^2 \\
&= 2\left(x_1 + \frac{1}{2}x_2 - x_3\right)^2 - \frac{3}{2}\left(x_2^2 + \frac{2}{3}x_2x_3 + \frac{1}{9}x_3^2\right) + \frac{1}{6}x_3^2 + x_3^2 \\
&= 2\left(x_1 + \frac{1}{2}x_2 - x_3\right)^2 - \frac{3}{2}\left(x_2 + \frac{1}{3}x_3\right)^2 + \frac{7}{6}x_3^2.
\end{aligned}$$

Substituting

$$\begin{aligned}
y_1 &= x_1 + \frac{1}{2}x_2 - x_3 \\
y_2 &= x_2 + \frac{1}{3}x_3 \\
y_3 &= x_3
\end{aligned}$$

we get the following canonical form of that quadratic form:

$$f(x) = 2y_1^2 - \frac{3}{2}y_2^2 + \frac{7}{6}y_3^2.$$

Example. Using the Lagrange's method find a canonical form of a quadratic form $f : \mathbb{R}^3 \to \mathbb{R}$ such that

$$f(x) = x_1x_2 - x_2x_3 + x_1x_3 \quad \text{for} \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3.$$

Solution.

Now, we don't have a term with x_i^2. So we put

$$x_1 = y_1 + y_2$$
$$x_2 = y_1 - y_2$$
$$x_3 = y_3$$

and get

$$\begin{aligned}
f(x) &= x_1x_2 - x_2x_3 + x_1x_3 \\
&= (y_1 + y_2)(y_1 - y_2) - (y_1 - y_2)y_3 + (y_1 + y_2)y_3 \\
&= y_1^2 - y_2^2 - y_1y_3 + y_2y_3 + y_1y_3 + y_2y_3 \\
&= y_1^2 - y_2^2 + 2y_2y_3 \\
&= y_1^2 - \left(y_2^2 - 2y_2y_3 + y_3^2\right) + y_3^2 \\
&= y_1^2 - (y_2 - y_3)^2 + y_3^2
\end{aligned}$$

Substituting

$$z_1 = y_1$$
$$z_2 = y_2 - y_3$$
$$z_3 = y_3$$

we get the following canonical form of that quadratic form:

$$f(x) = z_1^2 - z_2^2 + z_3^2.$$

Definition. $f : \mathbb{R}^n \to \mathbb{R}$ – a quadratic form

$$f \text{ is } \textit{positive definite} \underset{df}{\Leftrightarrow} f(x) > 0 \text{ for any } x \in \mathbb{R}^n$$

Theorem. $f : \mathbb{R}^n \to \mathbb{R}$ – a quadratic form, $f(x) = \sum\limits_{i=1}^{n} \delta_i x_i^2$ – a canonical form of f

Then f is positive definite iff $\delta_i > 0$ for all $i = 1, \ldots, n$.

Proof. Easy. $\square$

Theorem. $f : \mathbb{R}^n \to \mathbb{R}$ – a positive definite quadratic form

$f(x) = \sum\limits_{i,j=1}^{n} \alpha_{ij} x_i x_j$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$

Then a matrix $A = [\alpha_{ij}]$ of f satisfies

1) $\alpha_{ii} > 0$ for any $i = 1, \ldots, n$,
2) $\det(A) > 0$.

Proof. 1) Let $\mathcal{A} = (e_1, \ldots, e_n)$ be the standard basis of $\mathbb{R}^n$. Then

$$f(e_i) = \alpha_{ii} > 0 \quad \text{for any} \quad i = 1, \ldots, n,$$

because f is positive definite.

2) Let $\mathcal{A}$ be the standard basis of $\mathbb{R}^n$, $\mathcal{B}$ be a canonical basis of f and B be a matrix of f in $\mathcal{B}$.

If $f(x) = \sum\limits_{i=1}^{n} \delta_i x_i^2$ in $\mathcal{B}$, then

$$B = \begin{bmatrix} \delta_1 & 0 & \ldots & 0 \\ 0 & \delta_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \delta_n \end{bmatrix}$$

and $\delta_i > 0$ for all $i = 1, \ldots, n$. Now, if $Q = [I_{\mathbb{R}^n}]_{\mathcal{B}}^{\mathcal{A}}$, then

$$B = Q^T A Q.$$

Hence,

$$\det(B) = \det\left(Q^T\right) \cdot \det(A) \cdot \det(Q) = (\det(Q))^2 \cdot \det(A).$$

Since $\det(B) = \delta_1 \cdots \delta_n > 0$ and $(\det(Q))^2 > 0$, it follows that $\det(A) > 0$. $\square$

Theorem. (**Jacobi**) $\mathcal{B} = (v_1, \ldots, v_n)$ – a basis of $\mathbb{R}^n$
$f : \mathbb{R}^n \to \mathbb{R}$ – a quadratic form, $f(x) = \sum\limits_{i,j=1}^{n} \alpha_{ij} x_i x_j$ in $\mathcal{B}$
If

$$\Delta_k = \det \begin{bmatrix} \alpha_{11} & \alpha_{12} & \ldots & \alpha_{1k} \\ \alpha_{21} & \alpha_{22} & \ldots & \alpha_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{k1} & \alpha_{k2} & \ldots & \alpha_{kk} \end{bmatrix} \neq 0 \quad \text{for all} \quad k = 1, \ldots, n,$$

then there exists a basis $\mathcal{B}' = (u_1, \ldots, u_n)$ of $\mathbb{R}^n$ in which f has a form

$$f(x) = \frac{\Delta_0}{\Delta_1} y_1^2 + \frac{\Delta_1}{\Delta_2} y_2^2 + \ldots + \frac{\Delta_{n-1}}{\Delta_n} y_n^2,$$

where $\Delta_0 = 1$.
Moreover,

$$f \text{ is positive definite} \quad \Leftrightarrow \quad \bigwedge_{k=1,\ldots,n} \Delta_k > 0.$$

(without proof)

Example. Using the Jacobi's method find a canonical form of a quadratic form $f : \mathbb{R}^3 \to \mathbb{R}$ such that

$$f(x) = 2x_1^2 + 5x_2^2 + x_3^2 - 4x_1x_2 + 2x_1x_3 \quad \text{for} \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3.$$

Is the form f positive definite?

Solution.

Let $\mathcal{B}$ be the standard basis of $\mathbb{R}^3$. Then a matrix of f in $\mathcal{B}$ has the form

$$\begin{bmatrix} 2 & -2 & 1 \\ -2 & 5 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

and

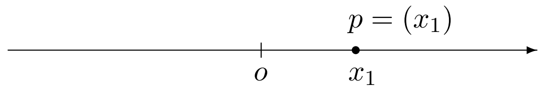
$$\Delta_0 = 1, \quad \Delta_1 = 2, \quad \Delta_2 = \begin{vmatrix} 2 & -2 \\ -2 & 5 \end{vmatrix} = 6, \quad \Delta_3 = \begin{vmatrix} 2 & -2 & 1 \\ -2 & 5 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 1.$$

Now, form f is positive definite, because $\Delta_1, \Delta_2, \Delta_3 > 0$ and f has the following canonical form:

$$f(x) = \frac{\Delta_0}{\Delta_1} y_1^2 + \frac{\Delta_1}{\Delta_2} y_2^2 + \frac{\Delta_2}{\Delta_3} y_3^2 = \frac{1}{2} y_1^2 + \frac{1}{3} y_2^2 + 6y_3^2.$$

3. Cartesian space $\mathbb{R}^n$

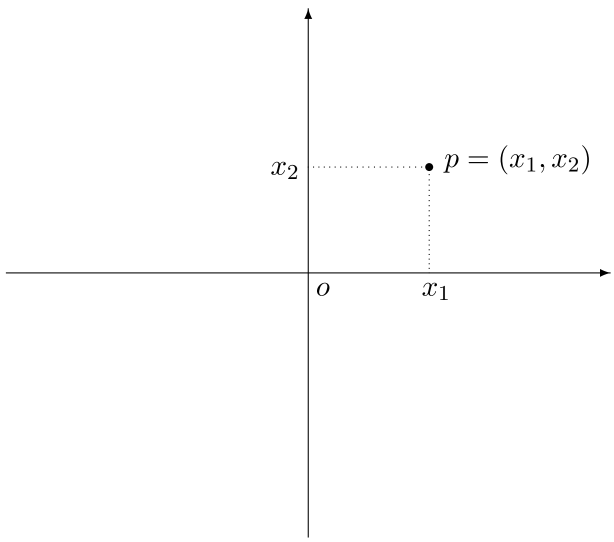
Cartesian coordinates on the line:



On the line we choose an arbitrary point o as the origin. It divides the line into two halflines. Regarding one of them as the positive halfline and the other as negative halfline, we obtain the axis. To any point p we assign a number x_1 called its Cartesian coordinate. In that way we get the **Cartesian space** $\mathbb{R}^1$.
The formula of the distance of two points $x, y \in \mathbb{R}^1$:

$$\rho(x,y) = |x-y|\,.$$

Cartesian coordinates on the plane:



On the plane let us consider two lines intersecting at a point o as the origin and on each of them let us fix Cartesian coordinates. We obtain the axes, which form the Cartesian system of coordinates.

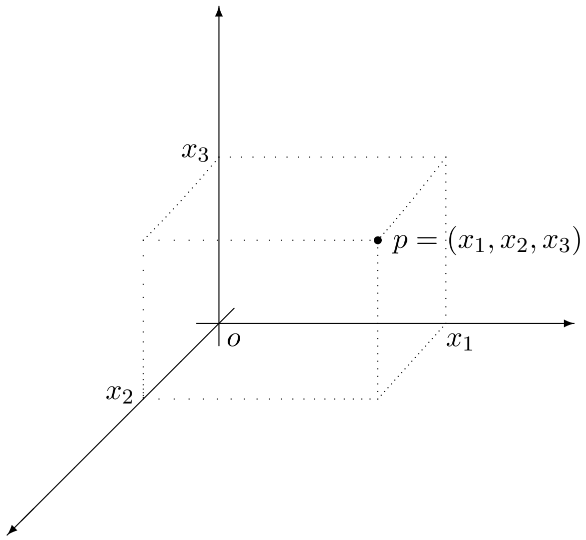
$p = (x_1, x_2)$ – Cartesian coordinates of the point p

If axes are perpendicular, then the Cartesian coordinates are called rectangular. In that way we get the **Cartesian space** $\mathbb{R}^2$.

The formula of the distance of two points $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$:

$$\rho(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

Cartesian coordinates in the space:



In the space let us take three lines not lying in one plane and passing through one point o as the origin, and on each of them let us fix Cartesian coordinates. We obtain the axes, which form the Cartesian system of coordinates.

$p = (x_1, x_2, x_3)$ – Cartesian coordinates of the point p

If each axis is perpendicular to both the remaining ones, then the system is called rectangular. In that way we get the **Cartesian space** $\mathbb{R}^3$.
The formula of the distance of two points $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathbb{R}^3$:

$$\rho(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}.$$

Definition. (**Metric space**) X – a set, $\rho : X \times X \to [0, \infty)$ – a function

A *metric space* is a pair (X, ρ) such that

1) $\bigwedge_{x,y \in X} \rho(x, y) = \rho(y, x)$,
2) $\bigwedge_{x,y \in X} \rho(x, y) = 0 \Leftrightarrow x = y$,
3) $\bigwedge_{x,y,z \in X} \rho(x, y) + \rho(y, z) \geq \rho(x, z)$.

Elements of X – points, ρ – a metrics, $\rho(x, y)$ – the distance of points x, y.

Definition. (**n-dimensional Cartesian space**) An n-dimensional Cartesian space is the set

$$\mathbb{R}^n = \{(x_1, \dots, x_n) : x_i \in \mathbb{R}\}$$

together with a metrics $\rho : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ given by formula

$$\rho((x_1, \dots, x_n), (y_1, \dots, y_n)) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

Thus $(\mathbb{R}^n, \rho)$ is a metric space.

Exercise. Show that a function ρ defined above is a metrics.

Definition. $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n$, $t \in \mathbb{R}$

Define

$$x + y \underset{df}{=} (x_1 + y_1, \dots, x_n + y_n) \quad - \quad \text{an addition of points } x, y,$$

$$-x \underset{df}{=} (-x_1, \dots, -x_n)$$

$$x - y \underset{df}{=} x + (-y) \quad - \quad \text{a subtraction of points } x, y,$$

$$tx \underset{df}{=} (tx_1, \dots, tx_n) \quad - \quad \text{a multiplication of a point } x \text{ by a number } t,$$

$$x \cdot y \underset{df}{=} \sum_{i=1}^{n} x_i y_i \quad - \quad \text{a scalar multiplication of points } x, y,$$

$$x^1 = x, x^{k+1} \underset{df}{=} x^k \cdot x \quad - \quad \text{a power of a point } x,$$

$$0 \underset{df}{=} (0, \dots, 0).$$

Theorem. $x, y, z \in \mathbb{R}^n$, $t \in \mathbb{R}$

We have

1) $x + y = y + x$,
2) $(x + y) + z = x + (y + z)$,
3) $t(x + y) = tx + ty$,
4) $tx = 0 \Leftrightarrow t = 0 \vee x = 0$,
5) $x \cdot y = y \cdot x$,
6) $\sim (x \cdot y) \cdot z = x \cdot (y \cdot z)$,
7) $(tx) \cdot y = t(x \cdot y)$,
8) $x \cdot (y + z) = x \cdot y + x \cdot z$,
9) $(tx)^k = t^k x^k$,
10) $\sim (x \cdot y)^k = x^k \cdot y^k$,
11) $(x \cdot y)^2 \leq x^2 \cdot y^2$ – Schwarz inequality.

Proof. Easy. $\square$

Definition. $x = (x_1, \dots, x_n) \in \mathbb{R}^n$

A *modulus* of a point x is a number:

$$|x| \underset{df}{=} \rho(x, 0) = \sqrt{\sum_{i=1}^{n} x_i^2}$$

(it is the distance of a point x and point 0).

Theorem. $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n$, $t \in \mathbb{R}$

We have

1) $x^2 = |x|^2 = \sum_{i=1}^{n} x_i^2$,
2) $\rho(x, y) = |x - y| = \sqrt{(x-y)^2}$,
3) $|x| \geq 0$,
4) $|x| = |-x|$,
5) $|x| = 0 \Leftrightarrow x = 0$,
6) $|tx| = |t|\,|x|$,
7) $|x \cdot y| \leq |x| \cdot |y|$,
8) $|x + y| \leq |x| + |y|$,
9) $|x| - |y| \leq |x - y|$,
10) $(x + y)^2 = x^2 + 2x \cdot y + y^2$,
11) $(x - y)^2 = x^2 - 2x \cdot y + y^2$,
12) $x^2 - y^2 = (x - y) \cdot (x + y)$.

Proof. 1) – 5) Easy.

6) $|tx| = \sqrt{\sum_{i=1}^{n} (tx_i)^2} = \sqrt{t^2 \sum_{i=1}^{n} x_i^2} = |t| \sqrt{\sum_{i=1}^{n} x_i^2} = |t|\,|x|$.

7) $|x \cdot y| = \sqrt{\sum_{i=1}^{n} (x_i y_i)^2} = \sqrt{\sum_{i=1}^{n} x_i^2 y_i^2} \leq \sqrt{\sum_{i=1}^{n} x_i^2 \cdot \sum_{i=1}^{n} y_i^2} = \sqrt{\sum_{i=1}^{n} x_i^2} \cdot \sqrt{\sum_{i=1}^{n} y_i^2} = |x| \cdot |y|$ (by Schwarz inequality).

8) $|x + y| = |x - (-y)| = \rho(x, -y) \leq \rho(x, 0) + \rho(0, -y) = \rho(x, 0) + \rho(0, y) = |x| + |y|$.

9) $|x| = |y + (x - y)| \leq |y| + |x - y|$, whence $|x| - |y| \leq |x - y|$.

10), 11) and 12) follow from 8) of previous theorem. □

Definition. (X, ρ) – a metric space, $a, b \in X$

A *metric segment* is a set:

$$\langle a, b \rangle \underset{df}{=} \{x \in X : \rho(a, x) + \rho(x, b) = \rho(a, b)\}.$$

Definition. (X, ρ) – a metric space, $a, b, c \in X$

$$c \text{ is a } \textit{centre} \text{ of a segment } \langle a, b \rangle \underset{df}{\Leftrightarrow} \rho(a, c) = \rho(b, c) = \frac{1}{2}\rho(a, b).$$

Theorem. $a, b \in \mathbb{R}^n$

There exists exactly one centre of a segment $\langle a, b\rangle$; it is a point $c = \frac{1}{2}(a+b)$.

Proof. If $a = b$, then Theorem is obvious. Let $a \neq b$. We have

$$\rho(a,c) = |a-c| = \left|a - \frac{1}{2}(a+b)\right| = \frac{1}{2}\,|a-b| = \frac{1}{2}\,|b-a| = \left|b - \frac{1}{2}(a+b)\right| = |b-c| = \rho(b,c).$$

Hence c is a centre of a segment $\langle a, b\rangle$.

Let $d = c + x$ be also a centre of a segment $\langle a, b\rangle$. Then

$$\rho(a,d) = \frac{1}{2}\rho(a,b) = \frac{1}{2}\,|a-b| = |a-d| = \left|a - \frac{1}{2}a - \frac{1}{2}b - x\right| = \left|\frac{1}{2}a - \frac{1}{2}b - \frac{1}{2}\cdot 2x\right| = \frac{1}{2}\,|a-b-2x|\,,$$

that is, $|a-b| = |a-b-2x|$.
Similarly,

$$\rho(b,d) = \frac{1}{2}\,|a-b| = |d-b| = \frac{1}{2}\,|a-b+2x|\,,$$

whence $|a-b| = |a-b+2x|$.
Thus,

$$|a-b-2x|^2 = |a-b+2x|^2\,,$$

that is,

$$(a-b)^2 - 4x(a-b) + 4x^2 = (a-b)^2 + 4x(a-b) + 4x^2,$$

whence

$$x(a-b) = 0.$$

Now, $a - b \neq 0$ (since $a \neq b$), so $x = 0$.
Thus, $d = c$. $\square$

Definition. $A \subseteq \mathbb{R}^n$

$$A \text{ is } \textit{convex} \underset{df}{\Leftrightarrow} \bigwedge_{a,b\in A} \langle a, b\rangle \subseteq A.$$

Conclusion. A segment in $\mathbb{R}^n$ is a convex set.

4. Vectors in space $\mathbb{R}^n$

Definition. A *localized vector* in $\mathbb{R}^n \underset{df}{=}$ an ordered pair of points in $\mathbb{R}^n$.

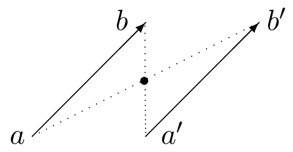
Denotation: $\overrightarrow{ab}$ for $a, b \in \mathbb{R}^n$, a – the initial point of $\overrightarrow{ab}$, b – the end-point of $\overrightarrow{ab}$.

Definition. Coordinates of a localized vector $\overrightarrow{ab} \underset{df}{=}$ coordinates of a point $b - a$.

If $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in \mathbb{R}^n$, then $\overrightarrow{ab} = [b_1 - a_1, \ldots, b_n - a_n]$.

Definition. $a, b, a', b' \in \mathbb{R}^n$

$$\overrightarrow{ab} = \overrightarrow{a'b'} \underset{df}{\Leftrightarrow} \overrightarrow{ab} \text{ and } \overrightarrow{a'b'} \text{ have the same coordinates } \underset{df}{\Leftrightarrow} b - a = b' - a'$$
$$\Leftrightarrow a' + b = a + b' \Leftrightarrow \frac{1}{2}(a' + b) = \frac{1}{2}(a + b')$$



(two localized vectors $\overrightarrow{ab}$ and $\overrightarrow{a'b'}$ are equal iff the centres of $\langle a', b\rangle$ and $\langle a, b'\rangle$ coinicide).

Theorem. The relation of equality of localized vectors is an equivalence relation.

Proof. Easy. □

Definition. A *free vector* (*vector*) in $\mathbb{R}^n \underset{df}{=}$ an equivalence class of the relation of equality of localized vectors,

that is,

$$\left[\overrightarrow{ab}\right] = \left\{\overrightarrow{cd} : \overrightarrow{ab} = \overrightarrow{cd}\right\} \quad - \quad \text{a free vector with a representative } \overrightarrow{ab}.$$

Denotation: $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \ldots$ (small gothic letters).

Remark. All representatives of a free vector have the same coordinates.

Definition. Coordinates of a free vector $\underset{df}{=}$ coordinates of its representative.

Definition. $\mathfrak{a}, a, b \in \mathbb{R}^n$, $\overrightarrow{ab} \in \mathfrak{a}$

$$|\mathfrak{a}| \underset{df}{=} \rho(a, b) \quad - \quad \text{a length of a vector } \mathfrak{a}.$$

If $\mathfrak{a} = [\alpha_1, \ldots, \alpha_n]$, then $|\mathfrak{a}| = \sqrt{\sum_{i=1}^n \alpha_i^2}$.

Definition. A *versor* $\underset{df}{=}$ a vector of length 1.

Theorem. (**On localization of a free vector at a point**) Every free vector in $\mathbb{R}^n$ can be uniquely localized at an arbitrary point $a \in \mathbb{R}^n$.

Proof. $\mathfrak{a}, a \in \mathbb{R}^n$
We search a point $b \in \mathbb{R}^n$ such that $\overrightarrow{ab} \in \mathfrak{a}$.
Let $\overrightarrow{cd} \in \mathfrak{a}$. Then

$$\overrightarrow{ab} = \overrightarrow{cd} \Leftrightarrow b - a = d - c \Leftrightarrow b = d - c + a. \quad \square$$

Theorem. For every free vector $\mathfrak{a} \in \mathbb{R}^n$ and every point $b \in \mathbb{R}^n$ there exists a unique representative of $\mathfrak{a}$ with the end-point b.

Proof. Similar (we calculate a). $\square$

Definition. $\mathfrak{a} = [\alpha_1, \ldots, \alpha_n], \mathfrak{b} = [\beta_1, \ldots, \beta_n] \in \mathbb{R}^n$, $t \in \mathbb{R}$

Define

$$\begin{aligned}
\mathfrak{a} + \mathfrak{b} &\underset{df}{=} [\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n] \quad - \quad \text{an addition of vectors } \mathfrak{a}, \mathfrak{b}, \\
-\mathfrak{a} &\underset{df}{=} [-\alpha_1, \ldots, -\alpha_n] \quad - \quad \text{an opposite vector for } \mathfrak{a}, \\
\mathfrak{a} - \mathfrak{b} &\underset{df}{=} [\alpha_1 - \beta_1, \ldots, \alpha_n - \beta_n] \quad - \quad \text{a subtraction of vectors } \mathfrak{a}, \mathfrak{b}, \\
t\mathfrak{a} &\underset{df}{=} [t\alpha_1, \ldots, t\alpha_n] \quad - \quad \text{a multiplication of a vector } \mathfrak{a} \text{ by a number } t, \\
\mathfrak{a} \cdot \mathfrak{b} &\underset{df}{=} \sum_{i=1}^{n} \alpha_i \beta_i \quad - \quad \text{a scalar product of vectors } \mathfrak{a}, \mathfrak{b}.
\end{aligned}$$

Remark. We will write $\mathfrak{a} \cdot \mathfrak{a} = \mathfrak{a}^2$.

Theorem. $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathbb{R}^n$, $t \in \mathbb{R}$

We have

1) $\mathfrak{a} \cdot \mathfrak{b} = \mathfrak{b} \cdot \mathfrak{a}$,
2) $(t\mathfrak{a}) \cdot \mathfrak{b} = t(\mathfrak{a} \cdot \mathfrak{b})$,
3) $\mathfrak{a}^2 = |\mathfrak{a}|^2$,
4) $\mathfrak{a} \cdot (\mathfrak{b} + \mathfrak{c}) = \mathfrak{a} \cdot \mathfrak{b} + \mathfrak{a} \cdot \mathfrak{c}$,
5) $-|\mathfrak{a}|\,|\mathfrak{b}| \leq \mathfrak{a} \cdot \mathfrak{b} \leq |\mathfrak{a}|\,|\mathfrak{b}|$.

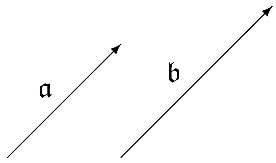
Proof. Easy. Point 5) follows from Schwartz inequality. $\square$

Theorem. $\overrightarrow{ab} \in \mathfrak{a} \ \wedge \ \overrightarrow{bc} \in \mathfrak{b} \ \Rightarrow \ \overrightarrow{ac} \in [\mathfrak{a} + \mathfrak{b}]$

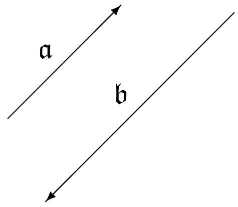
Proof. Easy. $\square$

Definition. $\mathfrak{a}, \mathfrak{b} \in \mathbb{R}^n$

$$\mathfrak{a}, \mathfrak{b} \text{ are } \textit{equally parallel}, \ \mathfrak{a} \upharpoonleft\upharpoonright \mathfrak{b} \underset{df}{\Leftrightarrow} |\mathfrak{a}| + |\mathfrak{b}| = |\mathfrak{a} + \mathfrak{b}|$$



$$\mathfrak{a}, \mathfrak{b} \text{ are } \textit{oppositely parallel}, \ \mathfrak{a} \upharpoonleft\downharpoonright \mathfrak{b} \underset{df}{\Leftrightarrow} |\mathfrak{a}| + |\mathfrak{b}| = |\mathfrak{a} - \mathfrak{b}|$$



$$\mathfrak{a}, \mathfrak{b} \text{ are } \textit{parallel}, \ \mathfrak{a} \parallel \mathfrak{b} \underset{df}{\Leftrightarrow} \mathfrak{a} \upharpoonleft\upharpoonright \mathfrak{b} \ \vee \ \mathfrak{a} \upharpoonleft\downharpoonright \mathfrak{b}$$

Theorem. $\mathfrak{a}, \mathfrak{b} \in \mathbb{R}^n$, $\mathfrak{a} \neq 0 \neq \mathfrak{b}$

Then,

$$\begin{aligned} \mathfrak{a} \parallel \mathfrak{b} \ &\Leftrightarrow \ \bigvee_{t \neq 0} \mathfrak{b} = t\mathfrak{a} \\ \text{and} \quad t > 0 \ &\Rightarrow \ \mathfrak{a} \upharpoonleft\upharpoonright \mathfrak{b}, \\ t < 0 \ &\Rightarrow \ \mathfrak{a} \upharpoonleft\downharpoonright \mathfrak{b}. \end{aligned}$$

Proof. Easy. $\square$

Theorem. In the set of nonzero vectors in $\mathbb{R}^n$ relations $\parallel$ and $\upharpoonleft\upharpoonright$ are equivalence relations.

Proof. Easy. $\square$

Definition. $\mathfrak{a} \in \mathbb{R}^n$

A *direction* of a vector $\mathfrak{a} \underset{df}{=}$ an equivalence class of the relation $\parallel$ with a representative $\mathfrak{a}$, that is,

$$\mathcal{K}(\mathfrak{a}) = \{\mathfrak{b} : \mathfrak{b} \parallel \mathfrak{a} \ \wedge \ \mathfrak{b} \neq 0\}.$$

A *sense* of a vector $\mathfrak{a} \underset{df}{=}$ an equivalence class of the relation $\upharpoonleft\upharpoonright$ with a representative $\mathfrak{a}$, that is,

$$\mathcal{Z}(\mathfrak{a}) = \{\mathfrak{b} : \mathfrak{b} \upharpoonleft\upharpoonright \mathfrak{a} \ \wedge \ \mathfrak{b} \neq 0\}.$$

We have: $\mathcal{Z}(\mathfrak{a}) \subseteq \mathcal{K}(\mathfrak{a})$.

Remark. $\mathfrak{a}, \mathfrak{b} \in \mathbb{R}^n$, $\mathfrak{a} \neq 0 \neq \mathfrak{b}$

Since $-|\mathfrak{a}|\,|\mathfrak{b}| \leq \mathfrak{a} \cdot \mathfrak{b} \leq |\mathfrak{a}|\,|\mathfrak{b}|$, it follows that there is a unique number θ such that

$$\mathfrak{a} \cdot \mathfrak{b} = |\mathfrak{a}|\,|\mathfrak{b}| \cos\theta \quad \text{and} \quad 0 \leq \theta \leq \pi.$$

If $\mathfrak{a} = 0$ or $\mathfrak{b} = 0$, then θ is arbitrary such that $0 \leq \theta \leq \pi$.

Definition. $\mathfrak{a}, \mathfrak{b} \in \mathbb{R}^n$

A number $\sphericalangle(\mathfrak{a}, \mathfrak{b}) \in [0, \pi]$ such that

$$\cos(\sphericalangle(\mathfrak{a}, \mathfrak{b})) = \frac{\mathfrak{a} \cdot \mathfrak{b}}{|\mathfrak{a}|\,|\mathfrak{b}|}$$

is called an *angle in* $\mathbb{R}^n$ *between vectors* $\mathfrak{a}, \mathfrak{b}$.

Theorem. $\mathfrak{a}, \mathfrak{b} \in \mathbb{R}^n$

We have

1) $\sphericalangle(\mathfrak{a}, \mathfrak{b}) = \sphericalangle(\mathfrak{b}, \mathfrak{a})$,
2) $t, s > 0 \;\Rightarrow\; \sphericalangle(\mathfrak{a}, \mathfrak{b}) = \sphericalangle(t\mathfrak{a}, s\mathfrak{b})$,
3) $\sphericalangle(\mathfrak{a}, \mathfrak{b}) + \sphericalangle(-\mathfrak{a}, \mathfrak{b}) = \pi$,
4) $\sphericalangle(\mathfrak{a}, \mathfrak{b}) = \sphericalangle(-\mathfrak{a}, -\mathfrak{b})$.

Proof. Easy. □

Definition. $\mathfrak{a}, \mathfrak{b} \in \mathbb{R}^n$

$$\mathfrak{a}, \mathfrak{b} \text{ are } \textit{perpendicular},\ \mathfrak{a} \perp \mathfrak{b} \underset{df}{\Leftrightarrow} \sphericalangle(\mathfrak{a}, \mathfrak{b}) = \frac{\pi}{2} \;\vee\; \mathfrak{a} = 0 \;\vee\; \mathfrak{b} = 0.$$

Theorem. $\mathfrak{a}, \mathfrak{b} \in \mathbb{R}^n$

Then

$$\mathfrak{a} \perp \mathfrak{b} \;\Leftrightarrow\; \mathfrak{a} \cdot \mathfrak{b} = 0.$$

Proof. Follows immediately from the formula $\mathfrak{a} \cdot \mathfrak{b} = |\mathfrak{a}|\,|\mathfrak{b}| \cos(\sphericalangle(\mathfrak{a}, \mathfrak{b}))$. □

Definition. (**Vector product in** $\mathbb{R}^3$) $\mathfrak{a}, \mathfrak{b} \in \mathbb{R}^3$, $\mathfrak{a} = [\alpha_1, \alpha_2, \alpha_3]$, $\mathfrak{b} = [\beta_1, \beta_2, \beta_3]$
A vector product of $\mathfrak{a}$ and $\mathfrak{b}$ is a vector

$$\mathfrak{a} \times \mathfrak{b} \underset{df}{=} \left[\begin{vmatrix} \alpha_2 & \alpha_3 \\ \beta_2 & \beta_3 \end{vmatrix}, - \begin{vmatrix} \alpha_1 & \alpha_3 \\ \beta_1 & \beta_3 \end{vmatrix}, \begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix} \right].$$

Remark. If we denote by i, j, k versors of coordinate axes in $\mathbb{R}^3$, that is, $i = [1,0,0]$, $j = [0,1,0]$ and $k = [0,0,1]$, then

$$\mathfrak{a} \times \mathfrak{b} = \begin{vmatrix} i & j & k \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{vmatrix}.$$

Example. Determine $\mathfrak{a} \times \mathfrak{b}$ if $\mathfrak{a} = [1,1,-1]$ and $\mathfrak{b} = [2,-1,3]$.

Solution.

$$\mathfrak{a} \times \mathfrak{b} = \begin{vmatrix} i & j & k \\ 1 & 1 & -1 \\ 2 & -1 & 3 \end{vmatrix} = \left[\begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix}, - \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} \right] = [2,-5,-3].$$

Theorem. $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathbb{R}^3$. Then

1) $\mathfrak{a} \times \mathfrak{a} = 0$,
2) $\mathfrak{a} \times \mathfrak{b} = -\mathfrak{b} \times \mathfrak{a}$,
3) $\mathfrak{a} \times (\mathfrak{b} + \mathfrak{c}) = \mathfrak{a} \times \mathfrak{b} + \mathfrak{a} \times \mathfrak{c}$ and $(\mathfrak{a} + \mathfrak{b}) \times \mathfrak{c} = \mathfrak{a} \times \mathfrak{c} + \mathfrak{b} \times \mathfrak{c}$,
4) $t \cdot (\mathfrak{a} \times \mathfrak{b}) = (t \cdot \mathfrak{a}) \times \mathfrak{b} = \mathfrak{a} \times (t \cdot \mathfrak{b})$, where $t \in \mathbb{R}$,
5) $(\mathfrak{a} \times \mathfrak{b}) \cdot \mathfrak{c} = \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix}$, where $\mathfrak{a} = [\alpha_1, \alpha_2, \alpha_3]$, $\mathfrak{b} = [\beta_1, \beta_2, \beta_3]$, $\mathfrak{c} = [\gamma_1, \gamma_2, \gamma_3]$,
6) $\mathfrak{a} \times \mathfrak{b} = 0 \Leftrightarrow \mathfrak{a} \parallel \mathfrak{b}$,
7) $\mathfrak{a} \times \mathfrak{b} \perp \mathfrak{a}$ and $\mathfrak{a} \times \mathfrak{b} \perp \mathfrak{b}$,
8) $|\mathfrak{a} \times \mathfrak{b}| = |\mathfrak{a}|\,|\mathfrak{b}| \sin \sphericalangle(\mathfrak{a}, \mathfrak{b})$.

Proof. Points 1) – 5) follow from above Remark.

6) We have

$$\mathfrak{a} \parallel \mathfrak{b} \;\Leftrightarrow\; \bigvee_{t \neq 0} \mathfrak{b} = t\mathfrak{a} \;\Leftrightarrow\; \bigvee_{t \neq 0} (t\mathfrak{a}) \times \mathfrak{b} = \mathfrak{b} \times \mathfrak{b} = 0 \;\Leftrightarrow\; \bigvee_{t \neq 0} t(\mathfrak{a} \times \mathfrak{b}) = 0 \;\Leftrightarrow\; \mathfrak{a} \times \mathfrak{b} = 0.$$

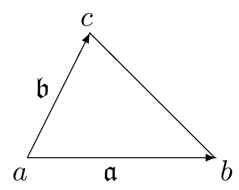
7) Follows from 5).

8) We have for $\mathfrak{a} = [\alpha_1, \alpha_2, \alpha_3]$ and $\mathfrak{b} = [\beta_1, \beta_2, \beta_3]$:

$$\begin{aligned} |\mathfrak{a} \times \mathfrak{b}|^2 &= (\alpha_2\beta_3 - \alpha_3\beta_2)^2 + (\alpha_1\beta_3 - \alpha_3\beta_1)^2 + (\alpha_1\beta_2 - \alpha_2\beta_1)^2 \\ &= (\alpha_1^2 + \alpha_2^2 + \alpha_3^2)(\beta_1^2 + \beta_2^2 + \beta_3^2) - (\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3)^2 \\ &= \mathfrak{a}^2\mathfrak{b}^2 - (\mathfrak{a} \cdot \mathfrak{b})^2 \\ &= (|\mathfrak{a}|\,|\mathfrak{b}|)^2 - (|\mathfrak{a}|\,|\mathfrak{b}|)^2 \cos^2 \sphericalangle(\mathfrak{a}, \mathfrak{b}) \\ &= (|\mathfrak{a}|\,|\mathfrak{b}| \sin \sphericalangle(\mathfrak{a}, \mathfrak{b}))^2, \end{aligned}$$

whence $|\mathfrak{a} \times \mathfrak{b}| = |\mathfrak{a}|\,|\mathfrak{b}| \sin \sphericalangle(\mathfrak{a}, \mathfrak{b})$. $\square$

Theorem. $a, b, c \in \mathbb{R}^3$, $\triangle(a,b,c)$ – a triangle with vertices a, b, c, $\mathfrak{a} = \left[\overrightarrow{ab}\right]$, $\mathfrak{b} = \left[\overrightarrow{ac}\right]$



Then

$$|\triangle(a,b,c)| = \frac{1}{2}\,|\mathfrak{a} \times \mathfrak{b}|$$

(the area of a triangle).

Proof. We have the following Heron's formula

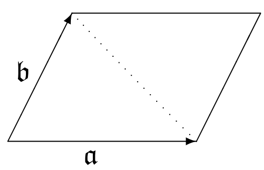
$$|\triangle(a,b,c)| = \frac{1}{4}\sqrt{s[s-2\rho(b,c)][s-2\rho(a,c)][s-2\rho(a,b)]},$$

where $s = \rho(a,b) + \rho(a,c) + \rho(b,c)$.
Hence

$$\begin{aligned}
|\triangle(a,b,c)| &= \frac{1}{4}\sqrt{(|\mathfrak{a}|+|\mathfrak{b}|+|\mathfrak{a}-\mathfrak{b}|)(|\mathfrak{a}|+|\mathfrak{b}|-|\mathfrak{a}-\mathfrak{b}|)(|\mathfrak{a}|-|\mathfrak{b}|+|\mathfrak{a}-\mathfrak{b}|)(-|\mathfrak{a}|+|\mathfrak{b}|+|\mathfrak{a}-\mathfrak{b}|)} \\
&= \frac{1}{2}\sqrt{(|\mathfrak{a}|\,|\mathfrak{b}| - \mathfrak{a}\cdot\mathfrak{b})(|\mathfrak{a}|\,|\mathfrak{b}| + \mathfrak{a}\cdot\mathfrak{b})} \\
&= \frac{1}{2}\sqrt{\mathfrak{a}^2\mathfrak{b}^2 - (\mathfrak{a}\cdot\mathfrak{b})^2} \\
&= \frac{1}{2}\,|\mathfrak{a}|\,|\mathfrak{b}|\sin\sphericalangle(\mathfrak{a},\mathfrak{b}).
\end{aligned}$$

Thus $|\triangle(a,b,c)| = \frac{1}{2}\,|\mathfrak{a}\times\mathfrak{b}|$. $\square$

Conclusion. The number $|\mathfrak{a}\times\mathfrak{b}|$ is the area of a parallelogram built on vectors $\mathfrak{a}$ and $\mathfrak{b}$:



Definition. (X,ρ), $(Y,\overline{\rho})$ – metric spaces, $f : X \to Y$ – a function

$$f \text{ is an } isometry \underset{df}{\Leftrightarrow} 1)\ f : X \overset{onto}{\longrightarrow} Y,$$
$$2) \bigwedge_{x,x' \in X} \overline{\rho}(f(x), f(x')) = \rho(x, x').$$

Examples.
1. Translation: $a \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(x) = x + a$ for $x \in \mathbb{R}^n$. Then f is an isometry, since

$$\rho(f(x), f(x')) = \sqrt{(f(x) - f(x'))^2} = \sqrt{[(x + a) - (x' + a)]^2} = \sqrt{(x - x')^2} = \rho(x, x')$$

for $x, x' \in \mathbb{R}^n$.
2. Rotation of the plane $\mathbb{R}^2$: $\alpha \in \mathbb{R}$, $x = (x_1, x_2) \in \mathbb{R}^2$, $f : \mathbb{R}^2 \to \mathbb{R}^2$
$f(x) = (x_1 \cos\alpha - x_2 \sin\alpha, x_1 \sin\alpha + x_2 \cos\alpha)$ – rotation through the angle α
Then f is an isometry, since

$$\begin{aligned}\rho(f(x), f(x'))^2 &= [(x_1 - x_1') \cos\alpha - (x_2 - x_2') \sin\alpha]^2 + [(x_1 - x_1') \sin\alpha + (x_2 - x_2') \cos\alpha]^2 \\ &= (x_1 - x_1')^2 + (x_2 - x_2')^2 \\ &= \rho(x, x')^2\end{aligned}$$

for $x = (x_1, x_2), x' = (x_1', x_2') \in \mathbb{R}^2$.

Theorem. An isometry is a one-to-one transformation.

Proof. (X,ρ), $(Y,\overline{\rho})$, $f : X \to Y$ – an isometry
Let $x, x' \in X$. Assume that $f(x) = f(x')$. Then

$$0 = \overline{\rho}(f(x), f(x')) = \rho(x, x') \ \Rightarrow\ x = x'. \ \square$$

Theorem. If $f : X \to Y$ is an isometry, then $f^{-1} : Y \to X$ is an isometry.

Proof. (X,ρ), $(Y,\overline{\rho})$, $f : X \to Y$ – an isometry
Obviously, f^{-1} is onto (because f is onto).
Let $y, y' \in Y$. There are $x, x' \in X$ such that $f^{-1}(y) = x$ and $f^{-1}(y') = x'$. Hence $y = f(x)$ and $y' = f(x')$. We have

$$\rho(f^{-1}(y), f^{-1}(y')) = \rho(x, x') = \overline{\rho}(f(x), f(x')) = \overline{\rho}(y, y'). \ \square$$

Theorem. Composition of two isometries is an isometry.

Proof. (X,ρ), $(Y,\overline{\rho})$, $(Z,\widehat{\rho})$, $f : X \to Y$, $g : Y \to Z$ – isometries

So

$$\bigwedge_{x,x'\in X} \overline{\rho}(f(x), f(x')) = \rho(x, x')$$

and

$$\bigwedge_{y,y'\in Y} \widehat{\rho}(g(y), g(y')) = \overline{\rho}(y, y').$$

Then $gf : X \to Z$ and

$$\bigwedge_{x,x'\in X} \widehat{\rho}(gf(x), gf(x')) = \overline{\rho}(f(x), f(x')) = \rho(x, x'). \ \Box$$

Definition. (X, ρ), $(Y, \overline{\rho})$ – metric spaces, $f : X \to Y$ – a function

$$f \text{ is a } \mathit{similarity} \underset{df}{\Leftrightarrow} 1)\ f : X \overset{onto}{\longrightarrow} Y,$$

$$2) \bigvee_{\lambda>0} \bigwedge_{x,x'\in X} \overline{\rho}(f(x), f(x') = \lambda\rho(x, x').$$

λ – the ratio of similarity

Remark. Any isometry is a similarity with the ratio 1.

Example. Homothety with the ratio $c > 0$: $j_c : \mathbb{R}^n \to \mathbb{R}^n$, $j_c(x) = cx$ for $x \in \mathbb{R}^n$. Then f is a similarity with the ratio c, since

$$\rho(j_c(x), j_c(x')) = \sqrt{(j_c(x) - j_c(x'))^2} = \sqrt{(cx - cx')^2} = c\sqrt{(x - x')^2} = c\rho(x, x')$$

for $x, x' \in \mathbb{R}^n$.

Theorem. A similarity is a one-to-one transformation.

Proof. (X, ρ), $(Y, \overline{\rho})$, $\quad f : X \to Y$ – a similarity with the ratio $\lambda > 0$
Let $x, x' \in X$ and $f(x) = f(x')$. Then

$$0 = \overline{\rho}(f(x), f(x')) = \lambda\rho(x, x')$$

and

$$\lambda > 0 \ \Rightarrow\ \rho(x, x') = 0 \ \Rightarrow\ x = x'. \ \Box$$

Theorem. If $f : X \to Y$ is a similarity with the ratio $\lambda > 0$, then $f^{-1} : Y \to X$ is a similarity with the ratio $\frac{1}{\lambda}$.

Proof. (X, ρ), $(Y, \overline{\rho})$, $\quad f : X \to Y$ – a similarity with the ratio $\lambda > 0$
Obviously, f^{-1} is onto (because f is onto).
Let $y, y' \in Y$. There are $x, x' \in X$ such that $f^{-1}(y) = x$ and $f^{-1}(y') = x'$. Hence $y = f(x)$ and $y' = f(x')$. We have

$$\rho(f^{-1}(y), f^{-1}(y')) = \rho(x, x') = \frac{1}{\lambda}\overline{\rho}(f(x), f(x')) = \frac{1}{\lambda}\overline{\rho}(y, y').$$

Thus f^{-1} is a similarity with the ratio $\frac{1}{\lambda}$. $\Box$

Theorem. Composition of two similarities is a similarity.

Proof. (X,ρ), $(Y,\overline{\rho})$, $(Z,\widehat{\rho})$

$f : X \to Y$ – a similarity with the ratio λ_1, $g : Y \to Z$ – a similarity with the ratio λ_2
We will show that $gf : X \to Z$ is a similarity with the ratio $\lambda_1\lambda_2$. Let $x, x' \in X$ and $y, y' \in Y$. We know that

$$\overline{\rho}(f(x), f(x')) = \lambda_1 \rho(x, x')$$

and

$$\widehat{\rho}(g(y), g(y')) = \lambda_2 \overline{\rho}(y, y').$$

We have

$$\widehat{\rho}(gf(x), gf(x')) = \lambda_2 \overline{\rho}(f(x), f(x')) = \lambda_1 \lambda_2 \rho(x, x'). \quad \square$$

Definition. (X,ρ), $(Y,\overline{\rho})$ – metric spaces

X and Y are *isometric* $\underset{df}{\Leftrightarrow}$ there exists an isometry $f : X \to Y$.

X and Y are *similar* $\underset{df}{\Leftrightarrow}$ there exists a similarity $g : X \to Y$.

Remark. If X, Y are isometric, then they are similar. The converse is not true.

6. Lines, planes and hyperplanes in space $\mathbb{R}^n$

Definition. (X,ρ) – a metric space, $Y \subseteq X$

$(Y, \rho|Y \times Y) \underset{df}{=}$ a subspace of a metric space (X,ρ).

Definition.

A *line* $\underset{df}{=}$ a subspace of the space $\mathbb{R}^n$ isometric with $\mathbb{R}^1$.

Remark. $L \subseteq \mathbb{R}^n$

L is a line $\Leftrightarrow$ L is isometric with $\mathbb{R}^1$ $\Leftrightarrow$ there exists an isometry $f : \mathbb{R}^1 \to L$ $\Leftrightarrow$ there exists an isometry $g : L \to \mathbb{R}^1$.

Remark. In $\mathbb{R}^1$ there exists a unique line. It is $\mathbb{R}^1$.

Theorem. (**On a line**) Through every two distinct points $a, b \in \mathbb{R}^n$ there passes exactly one line. It is the set $\{x(t) = (1-t)a + tb : t \in \mathbb{R}\} = L(a,b)$, where $x : \mathbb{R} \to \mathbb{R}^n$ is called the parametric presentation of a line $L(a,b)$.

Proof. Take $f : \mathbb{R}^1 \to L(a,b)$ such that $f(t) = x\left(\frac{t}{\rho(a,b)}\right)$, $t \in \mathbb{R}$. We have for $t, t' \in \mathbb{R}$:

$$\begin{aligned}\rho(f(t), f(t'))^2 &= \left[\left(1 - \frac{t}{\rho(a,b)}\right)a + \frac{t}{\rho(a,b)}b - \left(1 - \frac{t'}{\rho(a,b)}\right)a - \frac{t'}{\rho(a,b)}b\right]^2 \\ &= \left[\frac{(t-t')a - (t-t')b}{\rho(a,b)}\right]^2 = (t-t')^2 \\ &= \rho(t,t')^2.\end{aligned}$$

Hence f is an isometry, that is, $L(a,b)$ is a line. Moreover, $x(0) = a$ and $x(1) = b$ whence $a, b \in L(a,b)$.

Now we show that $L(a,b)$ is unique. Assume that there is a line K such that $a, b \in K$. We show that $K \subseteq L(a,b)$.

$g : \mathbb{R}^1 \to K$ – an isometry

There are $\alpha, \beta \in \mathbb{R}$ such that $g(\alpha) = a$, $g(\beta) = b$ and $\alpha < \beta$.

Take $c = g(\gamma) \in K$ such that $a \neq c \neq b$. Suppose that $\alpha < \beta < \gamma$. Then $|\beta - \alpha| + |\gamma - \beta| = |\gamma - \alpha|$. Hence $\rho(b,a) + \rho(c,b) = \rho(c,a)$, because g is an isometry. It follows

$$\left|\overrightarrow{ab}\right| + \left|\overrightarrow{bc}\right| = \left|\overrightarrow{ac}\right| = \left|\overrightarrow{ab} + \overrightarrow{bc}\right|,$$

so $\overrightarrow{ab} \parallel \overrightarrow{ac}$. Thus there exists $t \neq 0$ such that $c - a = t(b-a)$, whence $c = (1-t)a + tb = x(t) \in L(a,b)$.

Similarly when $\alpha < \gamma < \beta$ and $\gamma < \alpha < \beta$. Hence $K \subseteq L(a,b)$. Precisely, $K = L(a,b)$. $\square$

Remark. We will write the following parametric equation of $L(a,b)$:

$$L = L(a,b) : x(t) = (1-t)a + tb,\ t \in \mathbb{R}.$$

Definition. $\mathfrak{a}, a, b \in \mathbb{R}^n, \quad L \subseteq \mathbb{R}^n$ – a line

$\overrightarrow{ab}$ lies on $L \underset{df}{\Leftrightarrow} a, b \in L$.

$$\mathfrak{a} \parallel L \underset{df}{\Leftrightarrow} \bigvee_{\overrightarrow{ab}} \overrightarrow{ab} \in \mathfrak{a} \wedge \overrightarrow{ab} \text{ lies on } L \Leftrightarrow \bigvee_{a,b \in L} \overrightarrow{ab} \in \mathfrak{a}.$$

Definition. $\mathfrak{a} \in \mathbb{R}^n, \quad L \subseteq \mathbb{R}^n$ – a line

A *direction* of a line $L \underset{df}{=}$ a direction of a vector $\mathfrak{a} \parallel L$.

A *direction vector* of a line $L \underset{df}{=}$ a vector $\mathfrak{a} \parallel L$.

Theorem. (**The second form of the parametric equation of a line in** $\mathbb{R}^n$)

$\mathfrak{a}, a \in \mathbb{R}^n, \quad L \subseteq \mathbb{R}^n$ – a line

Then

$$a \in L \wedge \mathfrak{a} \parallel L \wedge \mathfrak{a} \neq 0 \Rightarrow L : x(t) = a + t\mathfrak{a},\ t \in \mathbb{R}.$$

Proof. $a \in L$, $\mathfrak{a} \parallel L$, $\mathfrak{a} \neq 0$

By Theorm on localization of a free vector at a point, a vector $\mathfrak{a}$ can be localized at a point a. Then there exists a point $b \in L$ (because $\mathfrak{a} \parallel L$) such that $\mathfrak{a} = \left[\overrightarrow{ab}\right]$.

By Theorem on a line for $t \in \mathbb{R}$:

$$\begin{aligned}
L &: x(t) = (1-t)a + tb, \text{ so} \\
L &: x(t) = a + t(b-a), \\
L &: x(t) = a + t\left[\overrightarrow{ab}\right], \\
L &: x(t) = a + t\mathfrak{a}. \quad \square
\end{aligned}$$

Remark. If $a = (a_1, \ldots, a_n) \in L$ and $\mathfrak{a} = [\alpha_1, \ldots, \alpha_n] \parallel L$, then a parametric equation of $L : x(t) = a + t\mathfrak{a},\ t \in \mathbb{R}$ has a form:

$$L : x(t) = (a_1 + t\alpha_1, \ldots, a_n + t\alpha_n),\ t \in \mathbb{R}.$$

For example, $L : x(t) = (1+2t, -1+3t),\ t \in \mathbb{R}$ is the line in $\mathbb{R}^2$ such that $a = (1,-1) \in L$ and $\mathfrak{a} = [2,3] \parallel L$, and $K : y(s) = (-1+s, 2-s, 3+2s),\ s \in \mathbb{R}$ is the line in $\mathbb{R}^3$ such that $a = (-1,2,3) \in K$ and $\mathfrak{a} = [1,-1,2] \parallel K$.

Definition. $L, K \subseteq \mathbb{R}^n$ – lines, $\mathfrak{a} \parallel L$, $\mathfrak{b} \parallel K$

$$L \parallel K \underset{df}{\Leftrightarrow} \mathfrak{a} \parallel \mathfrak{b} \Leftrightarrow \bigvee_{t \neq 0} \mathfrak{b} = t\mathfrak{a}.$$

$$L \perp K \underset{df}{\Leftrightarrow} \mathfrak{a} \perp \mathfrak{b} \Leftrightarrow \mathfrak{a} \cdot \mathfrak{b} = 0.$$

Definition. $\mathfrak{a} \in \mathbb{R}^2$, $L \subseteq \mathbb{R}^2$ – a line

A *normal direction* of a line $L \underset{df}{=}$ a direction of a vector $\mathfrak{a} \perp L$.

A *normal vector* of a line $L \underset{df}{=}$ a vector $\mathfrak{a} \perp L$.

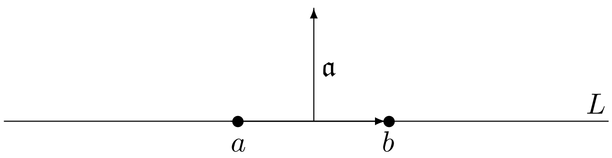
Theorem. For every point $a \in \mathbb{R}^2$ and every nonzero vector $\mathfrak{a} = [\alpha_1, \alpha_2]$ there exists in $\mathbb{R}^2$ a unique line, which passes through a with a normal vector $\mathfrak{a}$. It is consisted of all points (x_1, x_2) satisfying the equation

$$\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 = 0, \text{ where } \alpha_0 = -a \cdot (\mathfrak{a}).$$

That is the linear equation of a line L such that $a \in L$ and $\mathfrak{a} \perp L$.

Proof. $a = (a_1, a_2) \in L$, $\mathfrak{a} = [\alpha_1, \alpha_2] \perp L$, $b = (x_1, x_2) \in \mathbb{R}^2$

Then



$$\begin{aligned} b \in L \Leftrightarrow \left[\overrightarrow{ab}\right] \perp \mathfrak{a} \Leftrightarrow \left[\overrightarrow{ab}\right] \cdot \mathfrak{a} = 0 \\ \Leftrightarrow [x_1 - a_1, x_2 - a_2] \cdot [\alpha_1, \alpha_2] = 0 \\ \Leftrightarrow \alpha_1(x_1 - a_1) + \alpha_2(x_2 - a_2) = 0 \\ \Leftrightarrow -(a_1\alpha_1 + a_2\alpha_2) + \alpha_1 x_1 + \alpha_2 x_2 = 0. \end{aligned}$$

Setting

$$\alpha_0 = -(a_1\alpha_1 + a_2\alpha_2) = -a \cdot (\mathfrak{a})$$

we get

$$L : \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 = 0.$$

Obviously, such line is unique. □

Theorem. $L, K \subseteq \mathbb{R}^2$ – lines, $L : \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 = 0$, $K : \beta_0 + \beta_1 x_1 + \beta_2 x_2 = 0$

Then

$$K = L \Leftrightarrow \bigvee_{t \neq 0} \beta_i = t\alpha_i \text{ for } i = 0, 1, 2.$$

$$K \parallel L \Leftrightarrow \bigvee_{t \neq 0} \beta_i = t\alpha_i \text{ for } i = 1, 2.$$

Proof. Easy. $\square$

Definition. $L, K \subseteq \mathbb{R}^2$ – lines, $a \in \mathbb{R}^2$

$$\rho(a, L) \underset{df}{=} \rho(a, b), \text{ where } b \in K \cap L \text{ and } a \in K \perp L$$

(a distance of a point a and a line L in $\mathbb{R}^2$).

Theorem. $L \subseteq \mathbb{R}^2$ – a line, $L : \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 = 0$, $a = (a_1, a_2) \in \mathbb{R}^2$

Then

$$\rho(a, L) = \frac{|\alpha_0 + \alpha_1 a_1 + \alpha_2 a_2|}{\sqrt{\alpha_1^2 + \alpha_2^2}}.$$

Proof. $\mathfrak{a} = [\alpha_1, \alpha_2] \perp L$, $x = (x_1, x_2) \in \mathbb{R}^2$. Then $L : \alpha_0 + x \cdot (\mathfrak{a}) = 0$.

Take a line K such that $K : x(t) = a + t\mathfrak{a}$. Then $b \in K \cap L$, that is, $b = a + t'\mathfrak{a}$ and $\alpha_0 + b \cdot (\mathfrak{a}) = 0$, whence

$$\begin{aligned}\alpha_0 + (a + t'\mathfrak{a}) \cdot (\mathfrak{a}) &= 0\\ \alpha_0 + a \cdot (\mathfrak{a}) + t'\mathfrak{a}^2 &= 0\\ t'\mathfrak{a}^2 &= -\alpha_0 - a \cdot (\mathfrak{a})\\ t' &= -\frac{\alpha_0 + a \cdot (\mathfrak{a})}{\mathfrak{a}^2}.\end{aligned}$$

Hence $b = a - \frac{\alpha_0 + a \cdot (\mathfrak{a})}{\mathfrak{a}^2}\mathfrak{a}$ and

$$\begin{aligned}\rho(a, L) = \rho(a, b) &= |b - a|\\ &= \left| a - \frac{\alpha_0 + a \cdot (\mathfrak{a})}{\mathfrak{a}^2}\mathfrak{a} - a \right|\\ &= \frac{|\alpha_0 + a \cdot (\mathfrak{a})|}{|\mathfrak{a}|^2} |\mathfrak{a}|\\ &= \frac{|\alpha_0 + \alpha_1 a_1 + \alpha_2 a_2|}{\sqrt{\alpha_1^2 + \alpha_2^2}}. \quad \square\end{aligned}$$

Definition. An equation $\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 = 0$ of a line L in $\mathbb{R}^2$ is called *normalized* if $\mathfrak{a} = [\alpha_1, \alpha_2]$ is a versor (so $|\mathfrak{a}| = 1$).

Conclusion. If $\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 = 0$ is a normalized equation of a line L in $\mathbb{R}^2$ and $a = (a_1, a_2) \in \mathbb{R}^2$, then

$$\rho(a, L) = |\alpha_0 + \alpha_1 a_1 + \alpha_2 a_2|.$$

Theorem. Every line in $\mathbb{R}^2$ has a normalized equation.

Proof. Easy. $\square$

Theorem. $L(a,b) \subseteq \mathbb{R}^2$ – a line, $a = (a_1, a_2), b = (b_1, b_2) \in \mathbb{R}^2$, $a \neq b$

Then

$$L(a,b) : \begin{vmatrix} 1 & a_1 & a_2 \\ 1 & b_1 & b_2 \\ 1 & x_1 & x_2 \end{vmatrix} = 0.$$

Proof. $\left[\overrightarrow{ab}\right] = [b_1 - a_1, b_2 - a_2] \parallel L(a,b)$

It is easy to see that

$$[b_1 - a_1, b_2 - a_2] \cdot [-(b_2 - a_2), b_1 - a_1] = 0,$$

whence

$$[-(b_2 - a_2), b_1 - a_1] \perp L(a,b)$$

so

$$L(a,b) : -(a_1, a_2) \cdot [-(b_2 - a_2), b_1 - a_1] - (b_2 - a_2)x_1 + (b_1 - a_1)x_2 = 0.$$

Hence

$$L(a,b) : (a_2 x_1 + b_1 x_2 + a_1 b_2) - (b_2 x_1 + a_1 x_2 + a_2 b_1) = 0,$$

that is,

$$L(a,b) : \begin{vmatrix} 1 & a_1 & a_2 \\ 1 & b_1 & b_2 \\ 1 & x_1 & x_2 \end{vmatrix} = 0. \quad \square$$

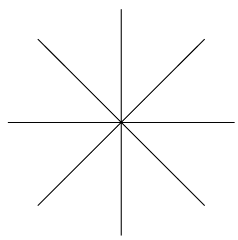
Remark. L, K – lines in $\mathbb{R}^2$

$L \parallel K \;\Rightarrow\; L = K \;\vee\; L \cap K = \emptyset,$

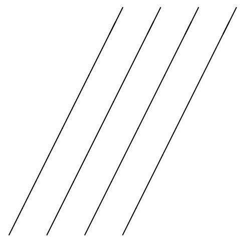
$L \nparallel K \;\Rightarrow\; L \cap K$ is a point.

Definition.

A *proper pencil of lines* in $\mathbb{R}^2 \underset{df}{=}$ the set of all lines which pass through one point



An *improper pencil of lines* in $\mathbb{R}^2 \underset{df}{=}$ the set of all lines with the same direction



Remark. Every two different lines in $\mathbb{R}^2$ determine a pencil (proper or improper). We use the following denotation:

$\mathrm{P}(L, K)$ = a pencil of lines in $\mathbb{R}^2$ determined by lines L, K.

Theorem. (**On a pencil of lines in** $\mathbb{R}^2$)

$L : \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 = 0$, $K : \beta_0 + \beta_1 x_1 + \beta_2 x_2 = 0$, $L \neq K$

Then

$$M \in \mathrm{P}(L, K) \Leftrightarrow \bigvee_{\eta,\lambda\in\mathbb{R},\, \eta^2+\lambda^2>0} M : \eta(\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2) + \lambda(\beta_0 + \beta_1 x_1 + \beta_2 x_2) = 0.$$

Proof. First, note that if $\eta^2 + \lambda^2 > 0$, then an equation

$$\eta(\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2) + \lambda(\beta_0 + \beta_1 x_1 + \beta_2 x_2) = 0$$

is a linear equation of some line in $\mathbb{R}^2$. Indeed, we have $[\alpha_1, \alpha_2] \neq 0 \neq [\beta_1, \beta_2]$, whence $[\eta\alpha_1 + \lambda\beta_1, \eta\alpha_2 + \lambda\beta_2] = \eta[\alpha_1, \alpha_2] + \lambda[\beta_1, \beta_2] \neq 0$.

($\Rightarrow$) $M \in \mathrm{P}(L, K)$, $a = (a_1, a_2) \in M$, $a \notin L \cup K$
It suffices to set: $\eta = \beta_0 + \beta_1 a_1 + \beta_2 a_2$ and $\lambda = -(\alpha_0 + \alpha_1 a_1 + \alpha_2 a_2)$.

($\Leftarrow$) Assume that

$$\bigvee_{\eta,\lambda\in\mathbb{R},\, \eta^2+\lambda^2>0} M : \eta(\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2) + \lambda(\beta_0 + \beta_1 x_1 + \beta_2 x_2) = 0.$$

We have two cases:

1) $\mathrm{P}(L, K)$ is proper.
Then an intersection point of lines L and K satisfies the equation of a line M, that is, $M \in \mathrm{P}(L, K)$.

2) $\mathrm{P}(L, K)$ is improper.
Then $\bigvee_{t\neq 0} [\beta_1, \beta_2] = t[\alpha_1, \alpha_2]$ (they are parallel), whence

$$\begin{aligned}[\eta\alpha_1 + \lambda\beta_1, \eta\alpha_2 + \lambda\beta_2] &= \eta[\alpha_1, \alpha_2] + \lambda[\beta_1, \beta_2] \\ &= \eta[\alpha_1, \alpha_2] + \lambda t[\alpha_1, \alpha_2] \\ &= (\eta + \lambda t)[\alpha_1, \alpha_2],\end{aligned}$$

that is, $M \parallel L \parallel K$. $\square$

Remark. Equivalently, we have

$$M \in \mathrm{P}(L,K) \Leftrightarrow \bigvee_{\lambda\in\mathbb{R}} M : \alpha_0+\alpha_1x_1+\alpha_2x_2+\lambda(\beta_0+\beta_1x_1+\beta_2x_2)=0$$

(in this case there does not exist λ such that $M = K$).

Definition.

Copenciled lines in $\mathbb{R}^2 \underset{df}{=}$ lines which belong to one pencil.

Theorem.

$L : \alpha_0+\alpha_1x_1+\alpha_2x_2 = 0$, $K : \beta_0+\beta_1x_1+\beta_2x_2 = 0$, $M : \gamma_0+\gamma_1x_1+\gamma_2x_2 = 0$ – different lines

Lines L, K, M are copenciled $\Leftrightarrow$

$$\begin{vmatrix} \alpha_0 & \beta_0 & \gamma_0 \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{vmatrix} = 0.$$

Proof. $M \in \mathrm{P}(L,K) \Leftrightarrow$ there are $\eta,\lambda,\delta\in\mathbb{R}$, $\eta^2+\lambda^2>0$ such that

$$\begin{cases} \eta\alpha_0+\lambda\beta_0 = -\delta\gamma_0, \\ \eta\alpha_1+\lambda\beta_1 = -\delta\gamma_1, \\ \eta\alpha_2+\lambda\beta_2 = -\delta\gamma_2, \end{cases}$$

which is equivalent to

$$\begin{cases} \eta\alpha_0+\lambda\beta_0+\delta\gamma_0 = 0, \\ \eta\alpha_1+\lambda\beta_1+\delta\gamma_1 = 0, \\ \eta\alpha_2+\lambda\beta_2+\delta\gamma_2 = 0. \end{cases}$$

That system has a nonzero solution $\Leftrightarrow$

$$\begin{vmatrix} \alpha_0 & \beta_0 & \gamma_0 \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{vmatrix} = 0. \qquad \square$$

Definition.

A *plane* $\underset{df}{=}$ a subspace of the space $\mathbb{R}^n$ isometric with $\mathbb{R}^2$.

Definition. $\mathfrak{a},\mathfrak{b},a,b\in\mathbb{R}^n$, $\quad P\subseteq\mathbb{R}^n$ – a plane

$\overrightarrow{ab}$ lies on $P \underset{df}{\Leftrightarrow} a,b\in P$.

$\mathfrak{a} \parallel P \underset{df}{\Leftrightarrow} \bigvee_{\overrightarrow{ab}} \overrightarrow{ab}\in\mathfrak{a} \wedge \overrightarrow{ab}$ lies on $P \Leftrightarrow \bigvee_{a,b\in P} \overrightarrow{ab}\in\mathfrak{a}$.

$\mathfrak{b}\perp P \underset{df}{\Leftrightarrow} \bigwedge_{\mathfrak{a}\parallel P} \mathfrak{b}\perp\mathfrak{a}$.

Definition. $P \subseteq \mathbb{R}^3$ – a plane, $\mathfrak{a} \in \mathbb{R}^3$

A *normal direction* of a plane $P \underset{df}{=}$ a direction of a vector $\mathfrak{a} \perp P$.

A *normal vector* of a plane $P \underset{df}{=}$ a vector $\mathfrak{a} \perp P$.

Definition. $P, Q \subseteq \mathbb{R}^3$ – planes, $\mathfrak{a}, \mathfrak{b} \in \mathbb{R}^3$

$P \parallel Q \underset{df}{\Leftrightarrow} \mathfrak{a} \perp P \ \wedge \ \mathfrak{b} \perp Q \ \wedge \ \mathfrak{a} \parallel \mathfrak{b}$.

$P \perp Q \underset{df}{\Leftrightarrow} \mathfrak{a} \perp P \ \wedge \ \mathfrak{b} \perp Q \ \wedge \ \mathfrak{a} \perp \mathfrak{b}$.

Theorem. For every point $a \in \mathbb{R}^3$ and every nonzero vector $\mathfrak{a} = [\alpha_1, \alpha_2, \alpha_3]$ there exists in $\mathbb{R}^3$ a unique plane, which passes through a with a normal vector $\mathfrak{a}$. It is consisted of all points (x_1, x_2, x_3) satisfying the equation

$$\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0, \text{ where } \alpha_0 = -a \cdot (\mathfrak{a}).$$

That is the linear equation of a plane P such that $a \in P$ and $\mathfrak{a} \perp P$.

Proof. Similar to the proof of theorem on a linear equation of a line. □

Theorem. $P, Q \subseteq \mathbb{R}^3$ – planes, $P : \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0$, $Q : \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 = 0$

Then

$P = Q \ \Leftrightarrow \ \bigvee_{t \neq 0} \beta_i = t\alpha_i$ for $i = 0, 1, 2, 3$.

$P \parallel Q \ \Leftrightarrow \ \bigvee_{t \neq 0} \beta_i = t\alpha_i$ for $i = 1, 2, 3$.

Proof. Easy. □

Definition. $P \subseteq \mathbb{R}^3$ – a plane, $L \subseteq \mathbb{R}^3$ – a line, $a \in \mathbb{R}^3$

$$\rho(a, P) \underset{df}{=} \rho(a, b), \text{ where } b \in P \cap L \text{ and } a \in L \perp P$$

(a distance of a point a and a plane P in $\mathbb{R}^3$).

Theorem. $P \subseteq \mathbb{R}^3$ – a plane, $P : \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0$, $a = (a_1, a_2, a_3) \in \mathbb{R}^3$

Then

$$\rho(a, P) = \frac{|\alpha_0 + \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3|}{\sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}}.$$

Proof. Similar to the proof of appropriate theorem for a line. □

Definition. An equation $\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0$ of a plane P in $\mathbb{R}^3$ is called *normalized* if $\mathfrak{a} = [\alpha_1, \alpha_2, \alpha_3]$ is a versor (so $|\mathfrak{a}| = 1$).

Conclusion. If $\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0$ is a normalized equation of a plane P in $\mathbb{R}^3$ and $a = (a_1, a_2, a_3) \in \mathbb{R}^3$, then

$$\rho(a, P) = |\alpha_0 + \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3|.$$

Theorem. Every plane in $\mathbb{R}^3$ has a normalized equation.

Proof. Easy. $\square$

Theorem. $P \subseteq \mathbb{R}^3$ – a plane, $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3), c = (c_1, c_2, c_3) \in \mathbb{R}^3$, $\overrightarrow{ab} \nparallel \overrightarrow{ac}$

Then

$$P : \begin{vmatrix} 1 & a_1 & a_2 & a_3 \\ 1 & b_1 & b_2 & b_3 \\ 1 & c_1 & c_2 & c_3 \\ 1 & x_1 & x_2 & x_3 \end{vmatrix} = 0.$$

Proof. Analogous to that for a line in $\mathbb{R}^2$. $\square$

Remark. $P, Q \subseteq \mathbb{R}^3$ – planes

$P \parallel Q \ \Rightarrow \ P = Q \ \vee \ P \cap Q = \emptyset$,

$P \nparallel Q \ \Rightarrow \ P \cap Q$ is a line.

Definition.

A *proper pencil* of planes in $\mathbb{R}^3 \underset{df}{=}$ the set of all planes containing the same line.

An *improper pencil* of planes in $\mathbb{R}^3 \underset{df}{=}$ the set of all planes with the same normal direction.

Remark. Every two different planes in $\mathbb{R}^3$ determine a pencil (proper or improper). We use the following denotation:

$\mathrm{P}(P, Q) =$ a pencil of planes in $\mathbb{R}^3$ determined by planes P, Q.

Theorem. (**On a pencil of planes in** $\mathbb{R}^3$)

$P : \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0,\ Q : \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 = 0,\ P \neq Q$

Then

$$R \in \mathrm{P}(P, Q) \Leftrightarrow \bigvee_{\eta, \lambda \in \mathbb{R},\, \eta^2 + \lambda^2 > 0} R : \eta(\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3) + \lambda(\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3) = 0.$$

Proof. Analogous to that for a pencil of lines in $\mathbb{R}^2$. $\square$

Remark. Equivalently, we have

$$R \in \mathrm{P}(P, Q) \Leftrightarrow \bigvee_{\lambda \in \mathbb{R}} R : \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \lambda(\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3) = 0$$

(in this case there does not exist λ such that $R = Q$).

Remark. $P, Q \subseteq \mathbb{R}^3$ – planes, $P \nparallel Q$

Then $P \cap Q = L$ is a line. If $P : \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0$, $Q : \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 = 0$, then

$$L : \begin{cases} \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0, \\ \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 = 0. \end{cases}$$

It is an *edge equation* of a line L in $\mathbb{R}^3$. Then $\mathfrak{a} = [\alpha_1, \alpha_2, \alpha_3] \perp L$ and $\mathfrak{b} = [\beta_1, \beta_2, \beta_3] \perp L$. Hence $\mathfrak{a} \times \mathfrak{b} \parallel L$.

Definition. $L \subseteq \mathbb{R}^3$ – a line, $P \subseteq \mathbb{R}^3$ – a plane, $a \in \mathbb{R}^3$

$$\rho(a, L) \underset{df}{=} \rho(a, b), \text{ where } b \in L \cap P \text{ and } a \in P \perp L$$

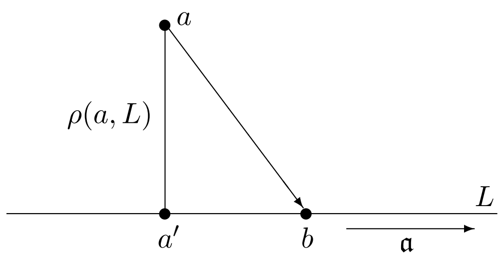
(a distance of a point a and a line L in $\mathbb{R}^3$).

Theorem. $L \subseteq \mathbb{R}^3$ – a line, $\mathfrak{a}, a, b \in \mathbb{R}^3$, $\mathfrak{a} \parallel L$, $a \neq b$, $b \in L$

Then

$$\rho(a, L) = \frac{\left| \mathfrak{a} \times \left[\overrightarrow{ab} \right] \right|}{|\mathfrak{a}|}.$$

Proof. We have



Hence $\sin\left(\sphericalangle \left(\mathfrak{a}, \left[\overrightarrow{ab} \right] \right) \right) = \frac{\rho(a,a')}{\rho(a,b)}$ and

$$\begin{aligned} \rho(a, L) = \rho(a, a') &= \rho(a, b) \sin\left(\sphericalangle \left(\mathfrak{a}, \left[\overrightarrow{ab} \right] \right) \right) \\ &= \frac{|\mathfrak{a}| \left| \left[\overrightarrow{ab} \right] \right| \sin\left(\sphericalangle \left(\mathfrak{a}, \left[\overrightarrow{ab} \right] \right) \right)}{|\mathfrak{a}|} \\ &= \frac{\left| \mathfrak{a} \times \left[\overrightarrow{ab} \right] \right|}{|\mathfrak{a}|}. \quad \square \end{aligned}$$

Definition. $k < n$

A *k-dimensional hyperplane* in $\mathbb{R}^n \underset{df}{=}$ a subspace of the space $\mathbb{R}^n$ isometric with $\mathbb{R}^k$.

Definition. $\mathfrak{a}, \mathfrak{b}, a, b \in \mathbb{R}^n$, H^{n-1} – an $(n-1)$-dimensional hyperplane in $\mathbb{R}^n$

$$\mathfrak{a} \parallel H^{n-1} \underset{df}{\Leftrightarrow} \bigvee_{\overrightarrow{ab}} \overrightarrow{ab} \in \mathfrak{a} \ \wedge \ a, b \in H^{n-1} \ \Leftrightarrow \bigvee_{a,b \in H^{n-1}} \overrightarrow{ab} \in \mathfrak{a}.$$

$$\mathfrak{b} \perp H^{n-1} \underset{df}{\Leftrightarrow} \bigwedge_{\mathfrak{a} \parallel H^{n-1}} \mathfrak{b} \perp \mathfrak{a}.$$

Theorem. For every point $a \in \mathbb{R}^n$ and every nonzero vector $\mathfrak{a} = [\alpha_1, \ldots, \alpha_n]$ there exists in $\mathbb{R}^n$ a unique hyperplane H^{n-1} such that $a \in H^{n-1}$ and $\mathfrak{a} \perp H^{n-1}$. It is consisted of all points $(x_1, \ldots, x_n)$ satisfying the equation

$$\alpha_0 + \alpha_1 x_1 + \ldots + \alpha_n x_n = 0, \text{ where } \alpha_0 = -a \cdot (\mathfrak{a}).$$

That is the linear equation of a hyperplane H^{n-1} such that $a \in H^{n-1}$ and $\mathfrak{a} \perp H^{n-1}$.

Proof. Similar to the proof of theorem on a linear equation of a line. $\square$

7. Transformations of space $\mathbb{R}^n$

Let $f:\mathbb{R}^n \to \mathbb{R}^n$ be an isometry, that is, f is onto and

$$\bigwedge_{x,y\in\mathbb{R}^n} \rho(f(x),f(y)) = \rho(x,y)$$

Definition.

An *invariant of isometry* $\underset{df}{=}$ a property which is unchanged by isometries.

Theorem. A centre of a segment is an invariant of isometry (that is, if c is a centre of a segment $\langle a,b\rangle$, then $f(c)$ is a centre of a segment $\langle f(a),f(b)\rangle$).

Proof. $f:\mathbb{R}^n\to\mathbb{R}^n$ – an isometry, $a,b,c\in\mathbb{R}^n$

If c is a centre of a segment $\langle a,b\rangle$, then

$$\rho(a,c)=\rho(b,c)=\frac{1}{2}\rho(a,b).$$

Hence

$$\rho(f(a),f(c))=\rho(f(b),f(c))=\frac{1}{2}\rho(f(a),f(b)),$$

that is, $f(c)$ is a centre of a segment $\langle f(a),f(b)\rangle$. $\square$

Theorem. An equality of localized vectors is an invariant of isometry.

Proof. Follows from definition of equal vectors and previous theorem. $\square$

Conclusion. $f:\mathbb{R}^n\to\mathbb{R}^n$ – an isometry, $\mathfrak{a},a,b\in\mathbb{R}^n$

Then

$$\overrightarrow{ab}\in\mathfrak{a} \;\Rightarrow\; f(\mathfrak{a})=\left[\overrightarrow{f(a)f(b)}\right].$$

Theorem. $f:\mathbb{R}^n\to\mathbb{R}^n$ – an isometry, $\mathfrak{a},\mathfrak{b}\in\mathbb{R}^n$

Then

1) $f(0)=0$ (for vectors!),
2) $f(\mathfrak{a}+\mathfrak{b})=f(\mathfrak{a})+f(\mathfrak{b})$,
3) $f(-\mathfrak{a})=-f(\mathfrak{a})$,
4) $f(\mathfrak{a}-\mathfrak{b})=f(\mathfrak{a})-f(\mathfrak{b})$,
5) $|f(\mathfrak{a})|=|\mathfrak{a}|$.

Proof. 1) Obvious.

2) $a,b,c\in\mathbb{R}^n$, $\overrightarrow{ab}\in\mathfrak{a}$ and $\overrightarrow{bc}\in\mathfrak{b}$ from theorem on localization of a free vector at a point

Then $\overrightarrow{ab}+\overrightarrow{bc}=\overrightarrow{ac}\in\mathfrak{a}+\mathfrak{b}$.

Hence $\overrightarrow{f(a)f(c)} \in f(\mathfrak{a}+\mathfrak{b})$ and $\overrightarrow{f(a)f(c)} = \overrightarrow{f(a)f(b)} + \overrightarrow{f(b)f(c)} \in f(\mathfrak{a}) + f(\mathfrak{b})$.

Thus $f(\mathfrak{a}+\mathfrak{b}) = f(\mathfrak{a}) + f(\mathfrak{b})$.

3) $0 = f(0) = f(\mathfrak{a} + (-\mathfrak{a})) = f(\mathfrak{a}) + f(-\mathfrak{a})$. Hence $f(-\mathfrak{a}) = -f(\mathfrak{a})$.

4) $f(\mathfrak{a}-\mathfrak{b}) = f(\mathfrak{a} + (-\mathfrak{b})) = f(\mathfrak{a}) + f(-\mathfrak{b}) = f(\mathfrak{a}) - f(\mathfrak{b})$.

5) $a, b \in \mathbb{R}^n$, $\overrightarrow{ab} \in \mathfrak{a}$

$$|f(\mathfrak{a})| = \left|\left[\overrightarrow{f(a)f(b)}\right]\right| = \rho(f(a), f(b)) = \rho(a,b) = \left|\left[\overrightarrow{ab}\right]\right| = |\mathfrak{a}|. \quad \square$$

Conclusion. The zero vector, an opposite vector, a sum and a difference of vectors and a length of a vector are invariants of isometry.

Theorem. Parallelism, equally parallelism and oppositely parallelism of vectors are invariants of isometry.

Proof. Follows from definition of parallelism and previous theorem. $\square$

Conclusion. A direction and a sense of a vector are invariants of isometry, that is, for $\mathfrak{a} \in \mathbb{R}^n$,

$$f(\mathcal{K}(\mathfrak{a})) = \mathcal{K}(f(\mathfrak{a})) \quad \text{and}$$
$$f(\mathcal{Z}(\mathfrak{a})) = \mathcal{Z}(f(\mathfrak{a})).$$

Theorem. $f : \mathbb{R}^n \to \mathbb{R}^n$ – an isometry, $\mathfrak{a} \in \mathbb{R}^n$, $t \in \mathbb{R}$

Then

$$f(t\mathfrak{a}) = tf(\mathfrak{a}).$$

Proof. Assume $t \geq 0$. Then $t\mathfrak{a} \upharpoonleft\upharpoonright \mathfrak{a}$, whence $f(t\mathfrak{a}) \upharpoonleft\upharpoonright f(\mathfrak{a})$ and $tf(\mathfrak{a}) \upharpoonleft\upharpoonright f(\mathfrak{a})$. Thus

$$f(t\mathfrak{a}) \upharpoonleft\upharpoonright tf(\mathfrak{a})$$

Moreover,

$$|f(t\mathfrak{a})| = |t\mathfrak{a}| = t\,|\mathfrak{a}| = t\,|f(\mathfrak{a})|\,.$$

Hence $f(t\mathfrak{a}) = tf(\mathfrak{a})$.

Similarly for $t < 0$ (in that case parallelism is opposite). $\square$

Conclusion. A linear combination of vectors is an invariant of isometry, that is,

$$f\left(\sum_{i=1}^{k} t_i\mathfrak{a}_i\right) = \sum_{i=1}^{k} t_i f(\mathfrak{a}_i),$$

where $\mathfrak{a}_1, \ldots, \mathfrak{a}_k \in \mathbb{R}^n$ and $t_1, \ldots, t_k \in \mathbb{R}$.

Theorem. A scalar product of vectors is an invariant of isometry, that is, $f(\mathfrak{a}) \cdot f(\mathfrak{b}) = \mathfrak{a} \cdot \mathfrak{b}$.

Proof. $f : \mathbb{R}^n \to \mathbb{R}^n$ – an isometry, $\mathfrak{a}, \mathfrak{b} \in \mathbb{R}^n$

We have

$$(f(\mathfrak{a})+f(\mathfrak{b}))^2 = (f(\mathfrak{a}+\mathfrak{b}))^2 = |f(\mathfrak{a}+\mathfrak{b})|^2 = |\mathfrak{a}+\mathfrak{b}|^2 = (\mathfrak{a}+\mathfrak{b})^2 = \mathfrak{a}^2+2\mathfrak{a}\cdot\mathfrak{b}+\mathfrak{b}^2 = |\mathfrak{a}|^2+2\mathfrak{a}\cdot\mathfrak{b}+|\mathfrak{b}|^2$$

and

$$(f(\mathfrak{a})+f(\mathfrak{b}))^2 = (f(\mathfrak{a}))^2+2f(\mathfrak{a})\cdot f(\mathfrak{b})+(f(\mathfrak{b}))^2 = |f(\mathfrak{a})|^2+2f(\mathfrak{a})\cdot f(\mathfrak{b})+|f(\mathfrak{b})|^2 = |\mathfrak{a}|^2+2f(\mathfrak{a})\cdot f(\mathfrak{b})+|\mathfrak{b}|^2 .$$

Hence $|\mathfrak{a}|^2 + 2\mathfrak{a}\cdot\mathfrak{b} + |\mathfrak{b}|^2 = |\mathfrak{a}|^2 + 2f(\mathfrak{a})\cdot f(\mathfrak{b}) + |\mathfrak{b}|^2$.

Thus

$$f(\mathfrak{a})\cdot f(\mathfrak{b}) = \mathfrak{a}\cdot\mathfrak{b}. \quad \square$$

Conclusion. A perpendicularity of vectors is an invariant of isometry.

Conclusion. A cosine of an angle between vectors and a measure of an angle between vectors are invariants of isometry.

Theorem. A k-dimensional hyperplane in $\mathbb{R}^n$ $(k < n)$ is an invariant of isometry, that is, if H^k is a k-dimensional hyperplane, then $f(H^k)$ is a k-dimensional hyperplane.

Proof. Follows from definition of a k-dimensional hyperplane and the fact that a composition of isometries is an isometry. $\square$

Conclusion. A line and a plane in $\mathbb{R}^n$ are invariants of isometry.

Conclusion. A pencil of lines in $\mathbb{R}^2$ and a pencil of planes in $\mathbb{R}^3$ are invariants of isometry.

Theorem. Parallelism and perpendicularity of lines in $\mathbb{R}^n$ and parallelism and perpendicularity of planes in $\mathbb{R}^3$ are invariants of isometry.

Proof. Follows from the fact that parallelism and perpendicularity of vectors are invariants of isometry. $\square$

Remark. Let us set:

$$\delta^i_j = \begin{cases} 0 & \text{if} \quad i \neq j, \\ 1 & \text{if} \quad i = j. \end{cases}$$

Theorem. (**On an analytic form of an isometry**) Every isometry $f : \mathbb{R}^n \to \mathbb{R}^n$ is a transformation given by a formula

$$f(x) = a + \sum_{i=1}^{n} x_i \cdot (\mathfrak{a}_i), \text{ where } \mathfrak{a}_i \cdot \mathfrak{a}_j = \delta^i_j.$$

Then $f(0) = a$ and $\mathfrak{a}_i = f(\mathfrak{e}_i)$, where $\mathfrak{e}_i = [\delta^i_1, \delta^i_2, \ldots, \delta^i_n]$, $i = 1, \ldots, n$.

Proof. First, note that $\mathfrak{e}_1 = [1, 0, 0, \ldots, 0], \mathfrak{e}_2 = [0, 1, 0, \ldots, 0], \ldots, \mathfrak{e}_n = [0, 0, 0, \ldots, 1]$. From properties of an isometry we know that an isometry is a linear transformation. Hence every isometry $f : \mathbb{R}^n \to \mathbb{R}^n$ is uniquely determined by its values $f(\mathfrak{e}_1), \ldots, f(\mathfrak{e}_n)$ in end-points of vectors $\mathfrak{e}_1, \ldots, \mathfrak{e}_n$.

Now, if $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, then $x = 0 + x_1\mathfrak{e}_1 + \ldots + x_n\mathfrak{e}_n$, whence $f(x) = f(0) + x_1 f(\mathfrak{e}_1) + \ldots + x_n f(\mathfrak{e}_n)$. Setting $f(0) = a$ and $f(\mathfrak{e}_i) = \mathfrak{a}_i$, $i = 1, \ldots, n$ we get $\mathfrak{a}_i \cdot \mathfrak{a}_j = \delta^i_j$ and

$$f(x) = a + \sum_{i=1}^{n} x_i \cdot (\mathfrak{a}_i). \quad \square$$

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a similarity with the ratio $\lambda > 0$, that is, f is onto and

$$\bigwedge_{x,y \in \mathbb{R}^n} \rho(f(x), f(y)) = \lambda \rho(x, y)$$

Definition.

A *similarity invariant* $\underset{df}{=}$ a property which is unchanged by similarities.

Remark. Every similarity invariant is an invariant of isometry (since an isometry is a similarity with the ratio 1). An invariant of isometry is a similarity invariant iff it does not depend on a distance of points in $\mathbb{R}^n$. Thus we have:

Theorem. Similarity invariants are: a centre of a segment, an equality of localized vectors, the zero vector, an opposite vector, a sum and a difference of vectors, a parallelism, an equally parallelism and an oppositely parallelism of vectors, a direction and a sense of a vector, a linear combination of vectors, a k-dimensional hyperplane in $\mathbb{R}^n$, a line in $\mathbb{R}^n$, a plane in $\mathbb{R}^n$, a parallelism and a perpendicularity of lines in $\mathbb{R}^n$ and a parallelism and a perpendicularity of planes in $\mathbb{R}^3$, a pencil of lines in $\mathbb{R}^2$ and a pencil of planes in $\mathbb{R}^3$.

Conclusion. $f : \mathbb{R}^n \to \mathbb{R}^n$ – a similarity, $\mathfrak{a}, a, b \in \mathbb{R}^n$

Then

$$\overrightarrow{ab} \in \mathfrak{a} \;\Rightarrow\; f(\mathfrak{a}) = \left[\overrightarrow{f(a)f(b)}\right].$$

Theorem. $f : \mathbb{R}^n \to \mathbb{R}^n$ – a similarity with the ratio $\lambda > 0$, $\mathfrak{a}, \mathfrak{b} \in \mathbb{R}^n$

Then

1) $|f(\mathfrak{a})| = \lambda\,|\mathfrak{a}|$,
2) $f(\mathfrak{a}) \cdot f(\mathfrak{b}) = \lambda^2(\mathfrak{a} \cdot \mathfrak{b})$.

Proof. 1) $a, b \in \mathbb{R}^n$, $\overrightarrow{ab} \in \mathfrak{a}$

$$|f(\mathfrak{a})| = \left|\left[\overrightarrow{f(a)f(b)}\right]\right| = \rho(f(a), f(b)) = \lambda\rho(a, b) = \lambda\left|\left[\overrightarrow{ab}\right]\right| = \lambda\,|\mathfrak{a}|\,.$$

2) $f(\mathfrak{a}) + f(\mathfrak{b}) = f(\mathfrak{a} + \mathfrak{b})$

Hence

$$(f(\mathfrak{a}) + f(\mathfrak{b}))^2 = (f(\mathfrak{a} + \mathfrak{b}))^2 = |f(\mathfrak{a} + \mathfrak{b})|^2 = \lambda^2\,|\mathfrak{a} + \mathfrak{b}|^2 = \lambda^2(\mathfrak{a} + \mathfrak{b})^2 = \lambda^2\,|\mathfrak{a}|^2 + 2\lambda^2\mathfrak{a} \cdot \mathfrak{b} + \lambda^2\,|\mathfrak{b}|^2$$

and

$$(f(\mathfrak{a})+f(\mathfrak{b}))^2=(f(\mathfrak{a}))^2+2f(\mathfrak{a})\cdot f(\mathfrak{b})+(f(\mathfrak{b}))^2$$
$$=|f(\mathfrak{a})|^2+2f(\mathfrak{a})\cdot f(\mathfrak{b})+|f(\mathfrak{b})|^2$$
$$=\lambda^2|\mathfrak{a}|^2+2f(\mathfrak{a})\cdot f(\mathfrak{b})+\lambda^2|\mathfrak{b}|^2.$$

Thus

$$f(\mathfrak{a})\cdot f(\mathfrak{b})=\lambda^2\mathfrak{a}\cdot\mathfrak{b}.\ \square$$

Conclusion. A length of a vector and a scalar product of vectors are not similarity invariants.

Theorem. A cosine of an angle between vectors is a similarity invariant.

Proof. $f:\mathbb{R}^n\to\mathbb{R}^n$ – a similarity with the ratio $\lambda>0$, $\mathfrak{a},\mathfrak{b}\in\mathbb{R}^n$

By previous theorem we have

$$f(\mathfrak{a})\cdot f(\mathfrak{b})=|f(\mathfrak{a})|\,|f(\mathfrak{b})|\cos(\sphericalangle(f(\mathfrak{a}),f(\mathfrak{b})))=\lambda^2|\mathfrak{a}|\,|\mathfrak{b}|\cos(\sphericalangle(f(\mathfrak{a}),f(\mathfrak{b})))$$

and

$$\lambda^2(\mathfrak{a}\cdot\mathfrak{b})=\lambda^2|\mathfrak{a}|\,|\mathfrak{b}|\cos(\sphericalangle(\mathfrak{a},\mathfrak{b})),$$

that is,

$$\cos(\sphericalangle(f(\mathfrak{a}),f(\mathfrak{b})))=\cos(\sphericalangle(\mathfrak{a},\mathfrak{b})).\ \square$$

Conclusion. A measure of an angle between vectors, in particular, a perpendicularity of vectors are similarity invariants.

Theorem. (**On an analytic form of a similarity**) Every similarity $f:\mathbb{R}^n\to\mathbb{R}^n$ with the ratio $\lambda>0$ is a transformation given by a formula

$$f(x)=a+\sum_{i=1}^n x_i\cdot(\mathfrak{a}_i),\ \text{where } \mathfrak{a}_i\cdot\mathfrak{a}_j=\lambda^2\delta^i_j.$$

Then $f(0)=a$ and $\mathfrak{a}_i=f(\mathfrak{e}_i)$, where $\mathfrak{e}_i=[\delta^i_1,\delta^i_2,\dots,\delta^i_n]$, $i=1,\dots,n$.

Proof. We have $g:\mathbb{R}^n\to\mathbb{R}^n$ such that $g(x)=\frac{1}{\lambda}f(x)$, where $x\in\mathbb{R}^n$, is an isometry, because

$$\rho(g(x),g(y))^2=[g(y)-g(x)]^2=\frac{1}{\lambda^2}[f(y)-f(x)]^2=\frac{1}{\lambda^2}\rho(f(x),f(y))^2=\rho(x,y)^2,$$

that is, $\rho(g(x),g(y))=\rho(x,y)$, where $x,y\in\mathbb{R}^n$.

By theorem on an analytic form of an isometry

$$g(x)=b+\sum_{i=1}^n x_i\cdot(\mathfrak{b}_i),$$

where $\mathfrak{b}_i\cdot\mathfrak{b}_j=\delta^i_j$, $g(0)=b$, $\mathfrak{b}_i=g(\mathfrak{e}_i)$ and $\mathfrak{e}_i=[\delta^i_1,\delta^i_2,\dots,\delta^i_n]$. Hence

$$f(x)=\lambda g(x)=\lambda b+\sum_{i=1}^n x_i\cdot(\lambda\mathfrak{b}_i).$$

Setting $a = \lambda b$ and $\mathfrak{a}_i = \lambda \mathfrak{b}_i$, $i = 1, \ldots, n$ we get

$$f(x) = a + \sum_{i=1}^{n} x_i \cdot (\mathfrak{a}_i)$$

and

$$\begin{aligned} \mathfrak{a}_i \cdot \mathfrak{a}_j &= (\lambda \mathfrak{b}_i) \cdot (\lambda \mathfrak{b}_j) = \lambda^2 (\mathfrak{b}_i \cdot \mathfrak{b}_j) = \lambda^2 \delta_j^i, \\ f(0) &= \lambda g(0) = \lambda b = a, \\ \mathfrak{a}_i &= \lambda \mathfrak{b}_i = \lambda g(\mathfrak{e}_i) = f(\mathfrak{e}_i), \end{aligned}$$

where $\mathfrak{e}_i = [\delta_1^i, \delta_2^i, \ldots, \delta_n^i]$, $i = 1, \ldots, n$. $\square$

Definition. $f : \mathbb{R}^n \to \mathbb{R}^n$

$$f \text{ is an } affine\ transformation \underset{df}{\Leftrightarrow} \begin{array}{l} 1)\ f : \mathbb{R}^n \overset{onto}{\underset{1-1}{\longrightarrow}} \mathbb{R}^n, \\ 2)\ \bigwedge\limits_{a,b,a',b' \in \mathbb{R}^n} \overrightarrow{ab} = \overrightarrow{a'b'} \Rightarrow \overrightarrow{f(a)f(b)} = \overrightarrow{f(a')f(b')}, \\ 3)\ \bigwedge\limits_{\mathfrak{a}_1, \mathfrak{a}_2 \in \mathbb{R}^n} \bigwedge\limits_{t_1, t_2 \in \mathbb{R}} f(t_1 \mathfrak{a}_1 + t_2 \mathfrak{a}_2) = t_1 f(\mathfrak{a}_1) + t_2 f(\mathfrak{a}_2). \end{array}$$

Conclusion. $f : \mathbb{R}^n \to \mathbb{R}^n$ – an affine transformation, $\mathfrak{a}, a, b \in \mathbb{R}^n$

Then

$$\overrightarrow{ab} \in \mathfrak{a} \Rightarrow f(\mathfrak{a}) = \left[\overrightarrow{f(a)f(b)} \right].$$

Conclusion. Every isometry and every similarity are affine transformations.

Definition. $\mathfrak{a}_1, \ldots, \mathfrak{a}_k \in \mathbb{R}^n$, $t_1, \ldots, t_k \in \mathbb{R}$

Vectors $\mathfrak{a}_1, \ldots, \mathfrak{a}_k$ are *linearly independent* $\underset{df}{\Leftrightarrow}$

$$\sum_{i=1}^{k} t_i \mathfrak{a}_i = 0 \Rightarrow t_1 = t_2 = \ldots = t_k = 0.$$

Theorem. (**On an analytic form of an affine transformation**) Every affine transformation $f : \mathbb{R}^n \to \mathbb{R}^n$ is given by a formula

$$f(x) = a + \sum_{i=1}^{n} x_i \cdot (\mathfrak{a}_i),$$

where vectors $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ are linearly independent. Then $f(0) = a$ and $\mathfrak{a}_i = f(\mathfrak{e}_i)$, where $\mathfrak{e}_i = [\delta_1^i, \delta_2^i, \ldots, \delta_n^i]$, $i = 1, \ldots, n$.

Proof. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we have $x = 0 + x_1 \cdot (\mathfrak{e}_1) + \ldots + x_n \cdot (\mathfrak{e}_n)$.

By definition of an affine transformation

$$f(x) = f(0) + x_1 \cdot f(\mathfrak{e}_1) + \ldots + x_n \cdot f(\mathfrak{e}_n).$$

Let us set: $f(0) = a$ and $f(\mathfrak{e}_i) = \mathfrak{a}_i$, $i = 1, \ldots, n$.

Then

$$f(x) = a + \sum_{i=1}^{n} x_i \cdot (\mathfrak{a}_i)$$

and from the fact that f is one-to-one:

$$f(x) = f(0) \;\Rightarrow\; x = 0,$$

$$\text{that is, } a + \sum_{i=1}^{n} x_i \cdot (\mathfrak{a}_i) = a \;\Rightarrow\; x_1 = \ldots = x_n = 0,$$

$$\text{so, } \sum_{i=1}^{n} x_i \cdot (\mathfrak{a}_i) = 0 \;\Rightarrow\; x_1 = \ldots = x_n = 0.$$

Hence vectors $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ are linearly independent. □

Theorem. Composition of two affine transformations is an affine transformation.

Proof. Easy. □

Theorem. If $f : \mathbb{R}^n \to \mathbb{R}^n$ is an affine transformation, then $f^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is an affine transformation.

Proof. Easy. □

Definition.

An *affine invariant* $\underset{df}{=}$ a property which is unchanged by affine transformations.

Conclusion. Affine invariants are: an equality of localized vectors, a linear combination of vectors and a parallelism of vectors.

Theorem. $f : \mathbb{R}^n \to \mathbb{R}^n$ – an affine transformation, $a, b \in \mathbb{R}^n$, $t \in \mathbb{R}$

Then

$$f((1-t)a + tb) = (1-t)f(a) + tf(b).$$

Proof. Easy. It suffices to use an analytic form of an affine transformation. □

Conclusion. A centre of a segment is an affine invariant.

Conclusion. A line in $\mathbb{R}^n$ is an affine invariant.

Conclusion. A plane in $\mathbb{R}^n$ and a k-dimensional hyperplane in $\mathbb{R}^n$ are affine invariants (because they are unions of lines).

Conclusion. A parallelism of lines in $\mathbb{R}^n$ and a parallelism of planes in $\mathbb{R}^3$ are affine invariants.

Remark. Every affine invariant is a similarity invariant (which means that if a property is not a similarity invariant, then it is not an affine invariant).

Conclusion. A length of a vector and a scalar product of vectors are not affine invariants.

Conclusion. A cosine of an angle between vectors, a measure of an angle between vectors, in particular, a perpendicularity of vectors are not affine invariants.

Conclusion. Every affine invariant is a similarity invariant and every similarity invariant is an invariant of isometry.

Definition. $A \in M_{n\times n}(\mathbb{R})$

A matrix A is called *orthogonal* $\underset{df}{\Leftrightarrow}$ columns of A are versors in $\mathbb{R}^n$ perpendicular to each other.

Theorem. $A \in M_{n\times n}(\mathbb{R})$

The following are equivalent:

1) A is orthogonal,
2) $A^T A = I$,
3) $A^{-1} = A^T$.

Proof. Easy. □

Conclusion. $A, B \in M_{n\times n}(\mathbb{R})$ – orthogonal matrices

Then

1) $\det(A) = \pm 1$,
2) A^T is orthogonal,
3) rows of A are versors in $\mathbb{R}^n$ perpendicular to each other,
4) A^{-1} is orthogonal,
5) AB is orthogonal.

Definition. $f : \mathbb{R}^n \to \mathbb{R}^n$ – an isometry (a similarity, an affine transformation)

$a = (a_{01}, \ldots, a_{0n}), \mathfrak{a}_i = [\alpha_{i1}, \ldots, \alpha_{in}] \in \mathbb{R}^n,\ \ i = 1, \ldots, n \ \ (x_1, \ldots, x_n), (\overline{x}_1, \ldots, \overline{x}_n) \in \mathbb{R}^n$

Then

$$f(x) = a + \sum_{i=1}^{n} x_i \cdot (\mathfrak{a}_i),$$

that is,

$$f(x_1, \ldots, x_n) = (\overline{x}_1, \ldots, \overline{x}_n) = (a_{01}, \ldots, a_{0n}) + x_1[\alpha_{11}, \ldots, \alpha_{1n}] + \ldots + x_n[\alpha_{n1}, \ldots, \alpha_{nn}],$$

so

$$\begin{cases} \overline{x}_1 = a_{01} + \alpha_{11}x_1 + \ldots + \alpha_{n1}x_n \\ \overline{x}_2 = a_{02} + \alpha_{12}x_1 + \ldots + \alpha_{n2}x_n \\ \quad\vdots \\ \overline{x}_n = a_{0n} + \alpha_{1n}x_1 + \ldots + \alpha_{nn}x_n \end{cases}$$

A matrix

$$A_f \underset{df}{=} \begin{bmatrix} \alpha_{11} & \ldots & \alpha_{n1} \\ \alpha_{12} & \ldots & \alpha_{n2} \\ \vdots & \ddots & \vdots \\ \alpha_{1n} & \ldots & \alpha_{nn} \end{bmatrix}$$

is called the *matrix of an isometry* (*a similarity*, *an affine transformation*) f.

Theorem. $f : \mathbb{R}^n \to \mathbb{R}^n$

A transformation f given by the above analytic formula is:

1) an affine transformation $\Leftrightarrow A_f$ is nonsingular,
2) a similarity with the ratio $\lambda > 0 \Leftrightarrow \frac{1}{\lambda}A_f$ is orthogonal,
3) an isometry $\Leftrightarrow A_f$ is orthogonal.

Proof. Follows from theorems on an analytic forms of these transformations. □

8. Algebraic sets in space $\mathbb{R}^n$

Definition. $\varphi : \mathbb{R}^n \to \mathbb{R}$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $i_1, \ldots, i_n \in \{0, \ldots, k\}$, $k \in \mathbb{N} \cup \{0\}$

φ is a *monomial in n variables* $\underset{df}{\Leftrightarrow}$ $\varphi(x) = \alpha_{i_1 \ldots i_n} x_1^{i_1} \ldots x_n^{i_n}$.

A *degree* of a monomial $\varphi \underset{df}{=} i_1 + \ldots + i_n$.

φ is a *polynomial in n variables* $\underset{df}{\Leftrightarrow}$ φ is a sum of monomials.

A *degree* of a polynomial $\varphi \underset{df}{=}$ the greatest of degrees of monomials occuring in a polynomial φ.

Example.
1. $\varphi(x) = 2x_1^2 x_2^3$ is the monomial of degree 5 in 2 variables.
2. $\varphi(x) = x_1^2 x_2 + 2x_2^2 x_3^2 - 3x_1 x_3 + 5x_1 - 4$ is the polynomial of degree 4 in 3 variables.

Definition. $\varphi : \mathbb{R}^n \to \mathbb{R}$ – a polynomial of degree k

An equation $\varphi(x) = 0$ is called the *algebraic equation* of degree k.

Definition. (**An algebraic set in** $\mathbb{R}^n$)

$\varphi : \mathbb{R}^n \to \mathbb{R}$ – a polynomial, $\varphi(x) = 0$ – an algebraic equation

An *algebraic set* $\underset{df}{=}$ a set of solutions of an algebraic equation,

that is, if $F \subseteq \mathbb{R}^n$, then

F is an algebraic set $\underset{df}{\Leftrightarrow}$ [there is a polynomial $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that $x \in F \Leftrightarrow \varphi(x) = 0$].

We will write $F : \varphi(x) = 0$.

A *degree* of a set $F \underset{df}{=}$ the least of degrees of algebraic equations describing a set F.

We denote it by $\deg(F)$.

Remarks.
1. Algebraic sets of degree 0 in $\mathbb{R}^n$: $\emptyset$ and $\mathbb{R}^n$ (since if a polynomial φ is of degree 0, then an equation $\varphi(x) = 0$ is either contradictory or it is an identity).
2. Algebraic sets of degree 1 in $\mathbb{R}^n$: $(n-1)$-dimensional hyperplanes (if $H^{n-1} : \alpha_0 + \alpha_1 x_1 + \ldots + \alpha_n x_n = 0$, then $\varphi(x_1, \ldots, x_n) = \alpha_0 + \alpha_1 x_1 + \ldots + \alpha_n x_n = 0$ is an algebraic equation of degree 1).
3. Algebraic sets of degree 2 in $\mathbb{R}^1$: 2-point sets (since a polynomial of degree 2 in one variable has at most 2 roots).
4. Algebraic sets of degree k in $\mathbb{R}^1$: k-point sets (since a polynomial of degree k in one variable has at most k roots).

Conclusion. A line in $\mathbb{R}^2$ and a plane in $\mathbb{R}^3$ are algebraic sets of degree 1.

Theorem. (**On position of a line under an algebraic set of degree** k)

$L, F \subseteq \mathbb{R}^n$, L – a line, F – an algebraic set of degree k

Then

$$L \subseteq F \quad \vee \quad \overline{\overline{L \cap F}} \leq k.$$

Proof. $F : \varphi(x) = 0$, φ – a polynomial of degree k

By theorem on a line: $L : x(t) = (1-t)a + tb$, where $t \in \mathbb{R}$ and $a, b \in L$, that is, $L : (x_1, \ldots, x_n) = a + (b-a)t$, where $t \in \mathbb{R}$ and $a, b \in L$.

We search all $t \in \mathbb{R}$ satisfying the following system of equations

$$\begin{cases} (x_1, \ldots, x_n) = a + (b-a)t, \\ \varphi(x_1, \ldots, x_n) = 0. \end{cases}$$

It is not difficult to see that there are no such t or all $t \in \mathbb{R}$ satisfy that system or at most k numbers t satisfy that system. Hence

$$L \cap F = \emptyset \quad \vee \quad L \cap F = L \quad \vee \quad \overline{\overline{L \cap F}} \leq \overline{\overline{\{t_1, \ldots, t_k\}}}.$$

Thus

$$L \subseteq F \quad \vee \quad \overline{\overline{L \cap F}} \leq k. \ \square$$

Definition.

An *transcendental set* $\underset{df}{=}$ a subset of $\mathbb{R}^n$ which is not an algebraic set of any degree.

Conclusion. If for a set $F \subseteq \mathbb{R}^n$ there exists a line L such that $L \cap F$ is a proper infinite subset of L, then the set F is transcendental.

Example. The sinusoid is a transcendental set.

Theorem. An algebraic set and its degree are affine invariants.

Proof. $F : \varphi(x) = 0$ – an algebraic set of degree k, $f : \mathbb{R}^n \to \mathbb{R}^n$ – an affine transformation

Then we know that f^{-1} is also an affine transformation. If $f(x_1, \ldots, x_n) = (\overline{x}_1, \ldots, \overline{x}_n)$, then $f^{-1}(\overline{x}_1, \ldots, \overline{x}_n) = (x_1, \ldots, x_n)$. From an analytic form of an affine transformation f^{-1} we have formulas for $x_1, \ldots, x_n$. We set them to the equation $\varphi(x_1, \ldots, x_n) = 0$ and obtain an algebraic equation of degree k of an algebraic set $\overline{F}$, that is, $f(F) = \overline{F}$. $\square$

Conclusion. An algebraic set and its degree are similarity invariants and also invariants of isometry.

Conclusion. A transcendental set is an affine invariant (so also a similarity invariant and an invariant of isometry).

Definition. $a, a' \in \mathbb{R}^n$, $H \subseteq \mathbb{R}^n$ – a hyperplane

a, a' are *symmetric with respect to* $H \underset{df}{\Leftrightarrow}$

$$c = \frac{a + a'}{2} \in H \ \wedge \ \left[\overrightarrow{aa'}\right] \perp H.$$

Definition. $F, H \subseteq \mathbb{R}^n$, F – an algebraic set, H – a hyperplane

H is a *hyperplane of symmetry of* $F \underset{df}{\Leftrightarrow}$

$$[a \in F \ \Rightarrow \ a' \in F, \text{ where } a' \text{ is symmetric to } a \text{ with respect to } H].$$

Remarks.
1. A 0-dimensional hyperplane of symmetry reduces to a point, called the *centre of symmetry* of the set F.
2. A 1-dimensional hyperplane of symmetry is a line, called the *axis of symmetry* of the set F.

Theorem. A centre of symmetry of an algebraic set is an affine invariant.

Proof. Follows directly from definition. □

Remark. An axis of symmetry of an algebraic set is not an affine invariant.

Algebraic sets of degree 2 in $\mathbb{R}^2$:

1. A 1-point set.

$a = (a_1, a_2), x = (x_1, x_2) \in \mathbb{R}^2$

Then

$$\{a\} : (x_1 - a_1)^2 + (x_2 - a_2)^2 = 0$$

and $\varphi(x) = (x_1 - a_1)^2 + (x_2 - a_2)^2$ is a polynomial of degree 2, that is, $\deg(\{a\}) = 2$.

2. A union of two different lines.

$L, K \subseteq \mathbb{R}^2$ – lines, $x = (x_1, x_2) \in \mathbb{R}^2$

$$L : \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 = 0, \quad K : \beta_0 + \beta_1 x_1 + \beta_2 x_2 = 0$$

Then

$$\begin{aligned} x \in L \cup K \ &\Leftrightarrow \ \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 = 0 \ \vee \ \beta_0 + \beta_1 x_1 + \beta_2 x_2 = 0 \\ &\Leftrightarrow \ (\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2)(\beta_0 + \beta_1 x_1 + \beta_2 x_2) = 0. \end{aligned}$$

So

$$L \cup K : (\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2)(\beta_0 + \beta_1 x_1 + \beta_2 x_2) = 0$$

and $\varphi(x) = (\alpha_0+\alpha_1x_1+\alpha_2x_2)(\beta_0+\beta_1x_1+\beta_2x_2)$ is a polynomial of degree 2, that is, $\deg(L\cup K) = 2$.

3. A circle.

$a = (a_1, a_2) \in \mathbb{R}^2$ – a centre, $r > 0$ – a radius, $x = (x_1, x_2) \in \mathbb{R}^2$

A circle is defined in the following way:

$$S = S(a,r) \underset{df}{=} \{x \in \mathbb{R}^2 : \rho(x,a) = r\}.$$

Hence

$$\begin{aligned} x \in S \;&\Leftrightarrow\; \rho(x,a) = r \;\Leftrightarrow\; [\rho(x,a)]^2 = r^2 \\ &\Leftrightarrow\; (x_1 - a_1)^2 + (x_2 - a_2)^2 = r^2. \end{aligned}$$

So

$$S : (x_1 - a_1)^2 + (x_2 - a_2)^2 - r^2 = 0$$

and $\varphi(x) = (x_1 - a_1)^2 + (x_2 - a_2)^2 - r^2$ is a polynomial of degree 2, that is, $\deg(S) = 2$.

4. A conic.

Definition. (**Conic**)

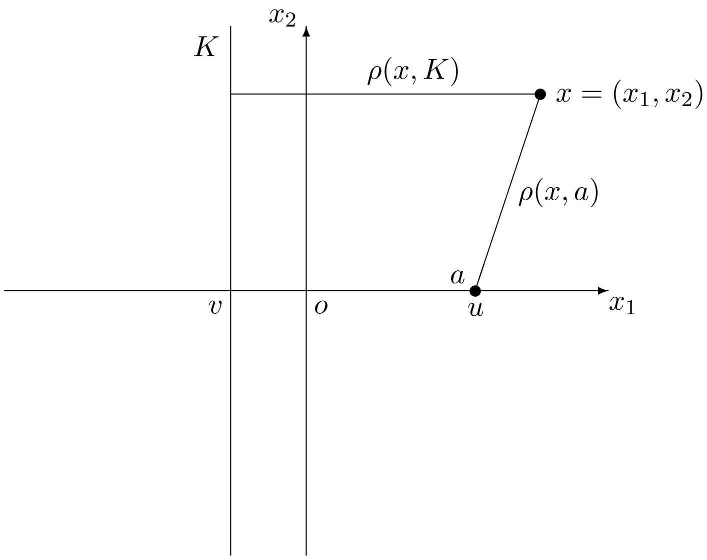
$a \in \mathbb{R}^2$, $K \subseteq \mathbb{R}^2$ – a line, $a \notin K$, $e > 0$

The set

$$S(a,K,e) \underset{df}{=} \{x \in \mathbb{R}^2 : \rho(x,a) = e \cdot \rho(x,K)\}$$

is called the *conic*. Then, a – a focus, K – a directrix, e – an eccentric.

Let us take such a coordinate system that the x_1-axis passes through the focus a and it is perpendicular to the directrix K, that is, $a = (u, 0)$, $K : x_1 - v = 0$ and $|u - v| = d$:



Then

$$\rho(x,a) = e \cdot \rho(x,K) \;\Leftrightarrow\; [\rho(x,a)]^2 = e^2 \cdot [\rho(x,K)]^2,$$

that is,

$$(x_1 - u)^2 + x_2^2 = e^2(x_1 - v)^2.$$

Hence

$$S(a,K,e) : (1-e^2)x_1^2 + x_2^2 + 2(e^2 v - u)x_1 + (u^2 - e^2 v^2) = 0$$

and $\varphi(x) = (1-e^2)x_1^2 + x_2^2 + 2(e^2 v - u)x_1 + (u^2 - e^2 v^2)$ is a polynomial of degree 2, that is, $\deg(S(a,K,e)) = 2$.

Theorem. A conic, its focus, directrix and eccentric are invariants of isometry.

Proof. $f : \mathbb{R}^2 \to \mathbb{R}^2$ – an isometry

$S(a,K,e) = \{x \in \mathbb{R}^2 : \rho(x,a) = e \cdot \rho(x,K)\}$ – a conic, a – a focus, K – a directrix, e – an eccentric

Then $f(K)$ is a line and

$$\rho(f(x), f(a)) = \rho(x,a) = e \cdot \rho(x,K) = e \cdot \rho(f(x), f(K)).$$

Hence

$$f(S(a,K,e)) = S(f(a), f(K), e) = \{y = f(x) \in \mathbb{R}^2 : \rho(y, f(a)) = e \cdot \rho(y, f(K))\}$$

is a conic which has a focus $f(a)$, a directrix $f(K)$ and an eccentric e. $\square$

Exercise. Show that a conic, its focus, directrix and eccentric are similarity invariants.

Definition.

$$\text{A conic } S(a,K,e) \text{ is : } \begin{aligned} &1) \text{ an } \textit{ellipse} \text{ if } e<1, \\ &2) \text{ a } \textit{parabola} \text{ if } e=1, \\ &3) \text{ a } \textit{hyperbola} \text{ if } e>1. \end{aligned}$$

We know that $a=(u,0)$, $K: x_1-v=0$, $|u-v|=d$ and

$$S(a,K,e):(1-e^2)x_1^2+x_2^2+2(e^2v-u)x_1+(u^2-e^2v^2)=0.$$

Parabola P:

$e=1$, let $u=\frac{1}{2}d$ and $v=-\frac{1}{2}d$

Then

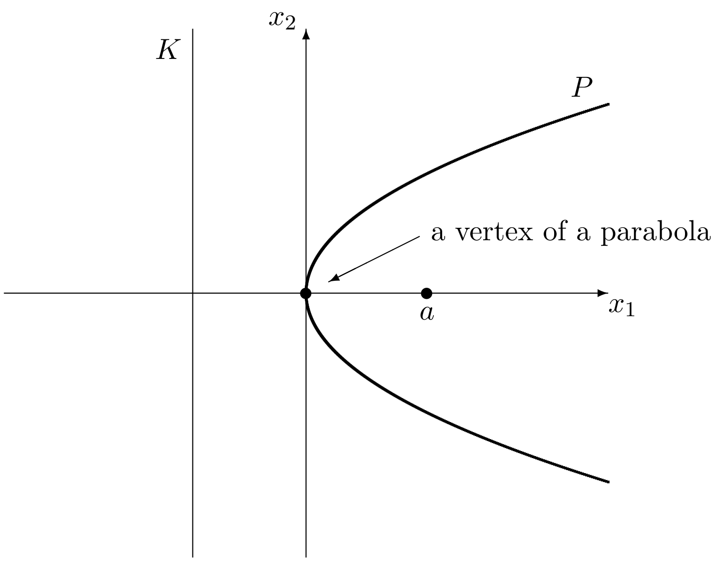
$$P: x_2^2+2(v-u)x_1+(u^2-v^2)=0,$$

that is,

$$P: x_2^2-2dx_1=0.$$

That is the *canonical equation of a parabola.*

It is easy to see that a parabola has one axis of symmetry: in canonical position the x_1-axis; does not have centres of symmetry; has a vertex, so a point of intersection of a parabola and its axis of symmetry: in canonical position point $(0,0)$; has one focus: in canonical position $a=(\frac{d}{2},0)$ and has one directrix: in canonical position $K: x_1+\frac{d}{2}=0$.



Ellipse E:

$e < 1$, let $v - u = d$ and $u - e^2 v = 0$

Hence

$$u = \frac{e^2 d}{1 - e^2}, \quad v = \frac{d}{1 - e^2} \quad \text{and} \quad u, v > 0.$$

Then

$$u^2 - e^2 v^2 = \frac{(e^2 d)^2}{(1 - e^2)^2} - \frac{e^2 d^2}{(1 - e^2)^2} = -ud.$$

Thus

$$E : \frac{(1 - e^2) x_1^2}{ud} + \frac{x_2^2}{ud} = 1.$$

Set: $\alpha_1 = \sqrt{\frac{ud}{1 - e^2}}$ and $\alpha_2 = \sqrt{ud}$, where

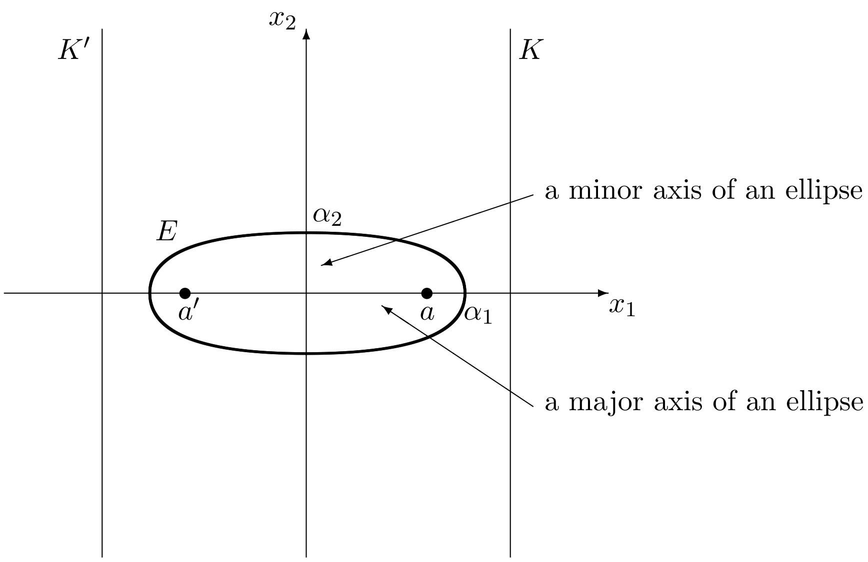
$$\alpha_1 = \frac{ed}{1 - e^2} > 0, \quad \alpha_2 = \frac{ed}{\sqrt{1 - e^2}} = \alpha_1 \sqrt{1 - e^2} < \alpha_1.$$

Then

$$E : \frac{x_1^2}{\alpha_1^2} + \frac{x_2^2}{\alpha_2^2} = 1.$$

That is the *canonical equation of an ellipse*.

It is easy to see that an ellipse has two axes of symmetry: in canonical position the coordinate axes; has one centre of symmetry: in canonical position point $(0, 0)$; has two foci: in canonical position $a = (\sqrt{\alpha_1^2 - \alpha_2^2}, 0)$ and $a' = (-\sqrt{\alpha_1^2 - \alpha_2^2}, 0)$ and has two directrices: in canonical position $K : x_1 - \frac{\alpha_1^2}{\sqrt{\alpha_1^2 - \alpha_2^2}} = 0$ and $K' : x_1 + \frac{\alpha_1^2}{\sqrt{\alpha_1^2 - \alpha_2^2}} = 0$. Moreover the eccentric $e = \frac{\sqrt{\alpha_1^2 - \alpha_2^2}}{\alpha_1}$.



Remark. A circle is an ellipse (with $\alpha_1 = \alpha_2$).

Hyperbola H**:**

$e > 1,$ let $v - u = d$ and $u - e^2 v = 0$

Hence

$$u = \frac{e^2 d}{1 - e^2}, \quad v = \frac{d}{1 - e^2} \quad \text{and} \quad u, v < 0.$$

Then

$$u^2 - e^2 v^2 = -ud.$$

Thus

$$H : \frac{(1 - e^2) x_1^2}{ud} + \frac{x_2^2}{ud} = 1.$$

Setting $\alpha_1 = \sqrt{\frac{ud}{1-e^2}}$ and $\alpha_2 = \sqrt{-ud}$, where

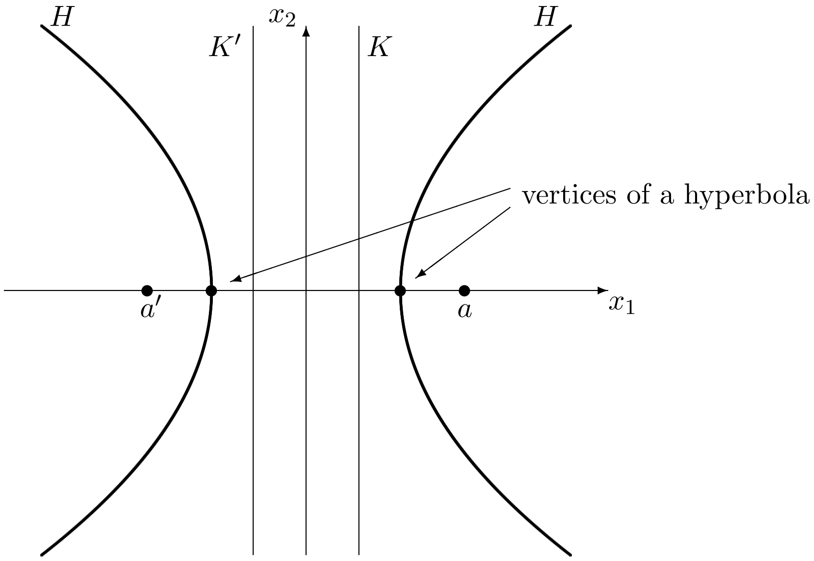
$$\alpha_1 = \frac{ed}{e^2 - 1} < -u, \quad \alpha_2 = \frac{ed}{\sqrt{e^2 - 1}} = \alpha_1 \sqrt{e^2 - 1} > \alpha_1,$$

we have

$$H : \frac{x_1^2}{\alpha_1^2} - \frac{x_2^2}{\alpha_2^2} = 1.$$

That is the *canonical equation of a hyperbola.*

It is easy to see that a hyperbola has two axes of symmetry: in canonical position the coordinate axes; has one centre of symmetry: in canonical position point $(0, 0)$; has two foci: in canonical position $a = (\sqrt{\alpha_1^2 + \alpha_2^2}, 0)$ and $a' = (-\sqrt{\alpha_1^2 + \alpha_2^2}, 0)$ and has two directrices: in canonical position $K : x_1 - \frac{\alpha_1^2}{\sqrt{\alpha_1^2 + \alpha_2^2}} = 0$ and $K' : x_1 + \frac{\alpha_1^2}{\sqrt{\alpha_1^2 + \alpha_2^2}} = 0$. Moreover the eccentric $e = \frac{\sqrt{\alpha_1^2 + \alpha_2^2}}{\alpha_1}$.



Theorem. All parabolas are similar.

Proof. Take a similarity $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$f(x) = \lambda x, \text{ where } \lambda > 0,$$

that is,

$$(\overline{x}_1, \overline{x}_2) = f(x_1, x_2) = (\lambda x_1, \lambda x_2).$$

Take a parabola $P : x_2^2 - 2dx_1 = 0$.

Then

$$(\lambda x_2)^2 - 2\lambda d \cdot (\lambda x_1) = 0.$$

Hence $P' : \overline{x}_2^2 - 2\lambda d\overline{x}_1 = 0$ and $\lambda d = d' \;\Rightarrow\; \lambda = \frac{d'}{d}$.

Thus the similarity f transforms the parabola P onto the parabola P'. $\square$

Theorem. All ellipses are identical from the affine point of view.

Proof. Take an affine transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$(\overline{x}_1, \overline{x}_2) = f(x_1, x_2) = (x_1, \sqrt{1 - e^2}\; x_2), \quad 0 < e < 1.$$

It is seen that f transforms the circle $S(0, \alpha_1) : x_1^2 + x_2^2 = \alpha_1^2$ onto the ellipse $E : \overline{x}_1^2 + \frac{\overline{x}_2^2}{1-e^2} = \alpha_1^2$, that is, onto the ellipse $E : \frac{\overline{x}_1^2}{\alpha_1^2} + \frac{\overline{x}_2^2}{\alpha_2^2} = 1$ (since $\alpha_2 = \alpha_1\sqrt{1 - e^2}$). Hence every ellipse is an affine image of the circle. Thus all ellipses are identical from the affine point of view. $\square$

Theorem. All hyperbolas are identical from the affine point of view.

Proof. Take an affine transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$(\overline{x}_1, \overline{x}_2) = f(x_1, x_2) = (\alpha_1 x_1, \alpha_2 x_2).$$

It is seen that f transforms the hyperbola $H_0 : x_1^2 - x_2^2 = 1$ onto the hyperbola $H : \frac{\overline{x}_1^2}{\alpha_1^2} - \frac{\overline{x}_2^2}{\alpha_2^2} = 1$. Hence every hyperbola is an affine image of the hyperbola H_0. Thus all hyperbolas are the same from the affine point of view. □

Algebraic sets of degree 2 in $\mathbb{R}^3$:

1. A 1-point set.

$a = (a_1, a_2, a_3), x = (x_1, x_2, x_3) \in \mathbb{R}^3$

Then

$$\{a\} : (x_1 - a_1)^2 + (x_2 - a_2)^2 + (x_3 - a_3)^2 = 0$$

and $\varphi(x) = (x_1 - a_1)^2 + (x_2 - a_2)^2 + (x_3 - a_3)^2$ is a polynomial of degree 2, that is, $\deg(\{a\}) = 2$.

2. A sphere.

$a = (a_1, a_2, a_3) \in \mathbb{R}^3$ – a centre, $r > 0$ – a radius, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$

A sphere is defined in the following way:

$$S = S(a, r) \underset{df}{=} \{x \in \mathbb{R}^3 : \rho(x, a) = r\}.$$

Hence

$$\begin{aligned} x \in S \;\Leftrightarrow\; \rho(x, a) = r \;&\Leftrightarrow\; [\rho(x, a)]^2 = r^2 \\ &\Leftrightarrow\; (x_1 - a_1)^2 + (x_2 - a_2)^2 + (x_3 - a_3)^2 = r^2. \end{aligned}$$

So

$$S : (x_1 - a_1)^2 + (x_2 - a_2)^2 + (x_3 - a_3)^2 - r^2 = 0$$

and $\varphi(x) = (x_1 - a_1)^2 + (x_2 - a_2)^2 + (x_3 - a_3)^2 - r^2$ is a polynomial of degree 2, that is, $\deg(S) = 2$.

3. A line.

$L \subseteq \mathbb{R}^3$ – a line, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$

$$L : \begin{cases} \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0, \\ \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 = 0. \end{cases}$$

Hence

$$L : (\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3)^2 + (\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3)^2 = 0$$

and $\varphi(x) = (\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3)^2 + (\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3)^2$ is a polynomial of degree 2, that is, $\deg(L) = 2$.

4. A union of two different planes.

$P, Q \subseteq \mathbb{R}^3$ – planes, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$

$$P : \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0, \quad Q : \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 = 0$$

Then

$$\begin{aligned} x \in P \cup Q \ &\Leftrightarrow \ \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0 \ \vee \ \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 = 0 \\ &\Leftrightarrow \ (\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3)(\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3) = 0. \end{aligned}$$

So

$$P \cup Q : (\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3)(\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3) = 0$$

and $\varphi(x) = (\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3)(\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3)$ is a polynomial of degree 2, that is, $\deg(P \cup Q) = 2$.

Definition. (**Set of revolution**)

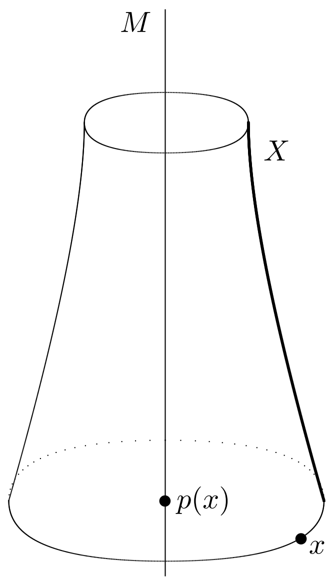
$M, X \subseteq \mathbb{R}^3$, M – a line, $P(x)$ – a plane such that $x \in P(x) \perp M$, $P(x) \cap M = p(x)$

$S(x) = \{y \in P(x) : \rho(y, p(x)) = \rho(x, p(x))\}$ – a circle in the plane $P(x)$ with centre $p(x)$ and passing through x

The set

$$S(X, M) \underset{df}{=} \bigcup_{x \in X} S(x)$$

is called the *set of revolution*. Then M is the axis of revolution.



Theorem. (**On an equation of a set of revolution**)

$$F \subseteq \mathbb{R}^3, \;\; F : \begin{cases} \varphi(x_2, x_3) = 0, \\ x_1 = 0, \end{cases} \qquad L_3 = x_3\text{-axis}$$

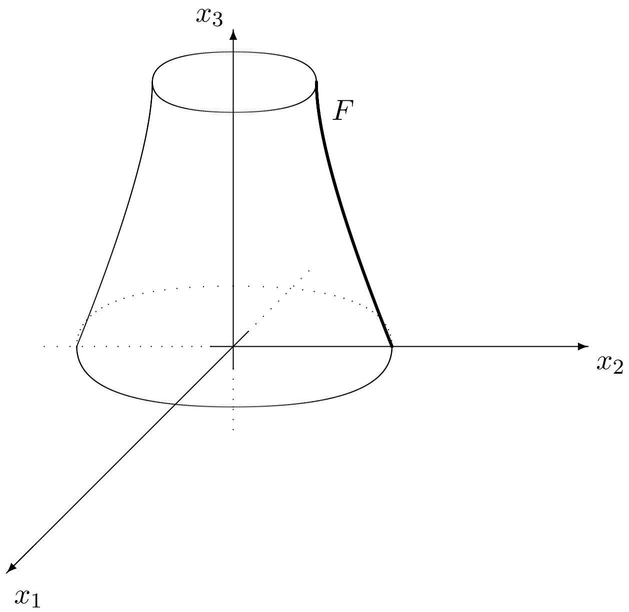
Then

$$S(F, L_3) : \varphi\left(\sqrt{x_1^2 + x_2^2}\,, x_3\right) \cdot \varphi\left(-\sqrt{x_1^2 + x_2^2}\,, x_3\right) = 0.$$

If L_3 is an axis of symmetry of F, then

$$S(F, L_3) : \varphi\left(\sqrt{x_1^2 + x_2^2}\,, x_3\right) = 0.$$

Proof. We have the situation:



Let $x = (x_1, x_2, x_3) \in \mathbb{R}^3, \;\; P : x_1 = 0.$

Then

$$x \in S(F, L_3) \Leftrightarrow S(x) \cap P = \left\{ y = \left(0, \sqrt{x_1^2 + x_2^2}\,, x_3\right), z = \left(0, -\sqrt{x_1^2 + x_2^2}\,, x_3\right) \wedge (y \in F \vee z \in F) \right\}.$$

Hence

$$\begin{aligned} x \in S(F, L_3) \;&\Leftrightarrow\; \varphi\left(\sqrt{x_1^2 + x_2^2}\,, x_3\right) = 0 \;\vee\; \varphi\left(-\sqrt{x_1^2 + x_2^2}\,, x_3\right) = 0 \\ &\Leftrightarrow\; \varphi\left(\sqrt{x_1^2 + x_2^2}\,, x_3\right) \cdot \varphi\left(-\sqrt{x_1^2 + x_2^2}\,, x_3\right) = 0. \end{aligned}$$

Thus

$$S(F, L_3) : \varphi\left(\sqrt{x_1^2 + x_2^2}\,, x_3\right) \cdot \varphi\left(-\sqrt{x_1^2 + x_2^2}\,, x_3\right) = 0.$$

If L_3 is an axis of symmetry of F, then

$$\varphi\left(\sqrt{x_1^2 + x_2^2}\,, x_3\right) = 0 \ \Leftrightarrow\ \varphi\left(-\sqrt{x_1^2 + x_2^2}\,, x_3\right) = 0.$$

Hence

$$S(F, L_3) : \varphi\left(\sqrt{x_1^2 + x_2^2}\,, x_3\right) = 0. \ \square$$

Cylinder of revolution:

Definition.

A *cylinder of revolution* $\underset{df}{=}$ a set built by revolution of a line about a line parallel to it (and different).

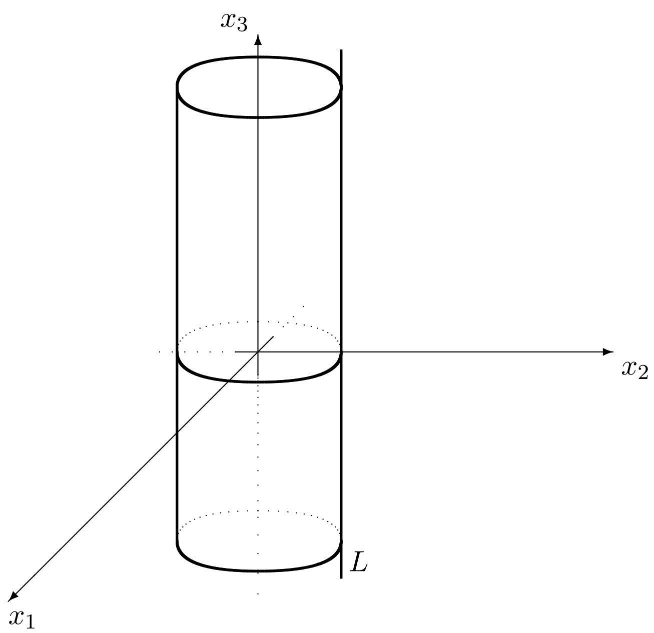
Let $r > 0$ and $L_3 = x_3$-axis. Take a line

$$L : \begin{cases} x_2 = r, \\ x_1 = 0, \end{cases}$$

that is,

$$L : \begin{cases} \varphi(x_2, x_3) = x_2 - r = 0, \\ x_1 = 0 \end{cases}$$

and revolve it about the axis L_3:



From theorem on an equation of a set of revolution we have

$$W = S(L, L_3) : \left(\sqrt{x_1^2 + x_2^2} - r\right)\left(-\sqrt{x_1^2 + x_2^2} - r\right) = 0, \text{ that is,}$$
$$W : x_1^2 + x_2^2 - r^2 = 0.$$

Hence

$$W : x_1^2 + x_2^2 = r^2.$$

That is the *canonical equation of a cylinder of revolution.* Then L is called a *rectilinear generator* of a cylinder.

Remark. Above equation is an equation of a circle lying in the plane $P : x_3 = 0$. Therefore we can define a cylinder of revolution in the following way.

A *cylinder of revolution* $\underset{df}{=}$ a union of all lines (rectilinear generators) intersecting this circle and perpendicular to P.

Definition. (**Cylinder over a planar set**)

$P, F \subseteq \mathbb{R}^3$, P – a plane, $F \subseteq P$

A *cylinder over* F $\underset{df}{=}$ a union of all lines (rectilinear generators) intersecting F and perpendicular to P.

Thus:

A cylinder of revolution = a cylinder over a circle.

An *elliptic cylinder* $\underset{df}{=}$ a cylinder over an ellipse.

A *parabolic cylinder* $\underset{df}{=}$ a cylinder over a parabola.

A *hyperbolic cylinder* $\underset{df}{=}$ a cylinder over a hyperbola.

Theorem. (**On an equation of a cylinder over a planar set**)

$F \subseteq \mathbb{R}^3$, $F : \begin{cases} \varphi(x_1, x_2) = 0, \\ x_3 = 0. \end{cases}$

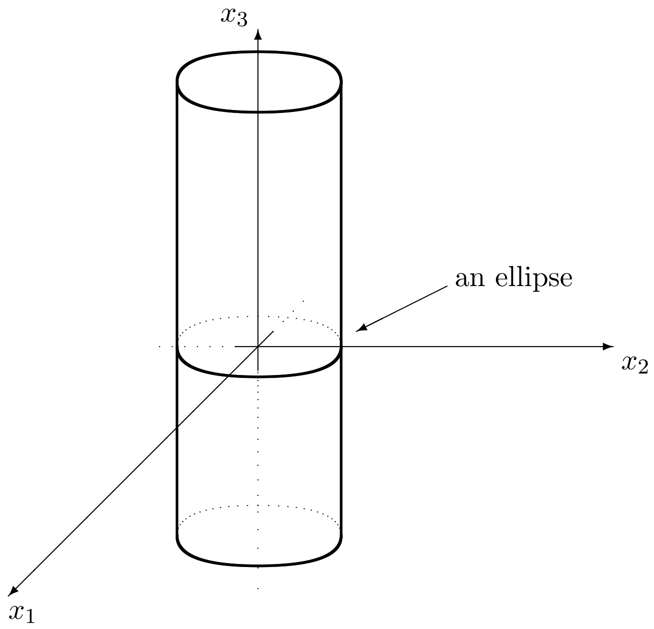
Then a cylinder over F has an equation:

$$WF : \varphi(x_1, x_2) = 0.$$

Proof. Obvious. □

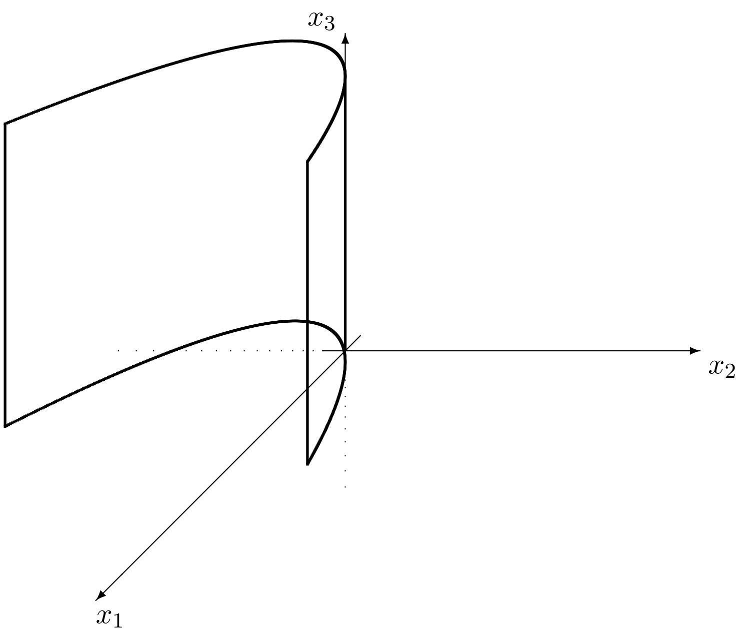
An elliptic cylinder WE:

$$WE : \frac{x_1^2}{\alpha_1^2} + \frac{x_2^2}{\alpha_2^2} = 1 \ \text{ – the canonical equation}$$



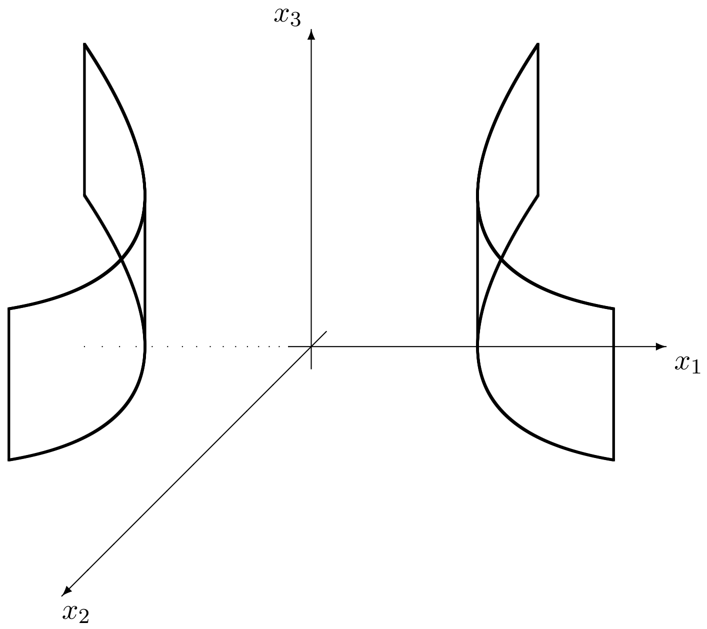
A parabolic cylinder WP:

$$WP : x_2^2 = 2dx_1 \ - \ \text{the canonical equation}$$



A hyperbolic cylinder WH:

$$WH : \frac{x_1^2}{\alpha_1^2} - \frac{x_2^2}{\alpha_2^2} = 1 \ - \ \text{the canonical equation}$$



Theorem. (**On generators of a cylinder**)

If W is a cylinder of revolution (or an elliptic cylinder or a parabolic cylinder or a hyperbolic cylinder), then through every $x \in W$ passes exactly one rectilinear generator of the cylinder W.

Proof. W – a cylinder in canonical position, $x \in W$, L – a generator of W such that $x \in L$

$P : x_3 = 0$, $P \cap W =$ a circle or a conic

Suppose that there is a generator L' of W such that $x \in L'$ and $L' \neq L$. Let

$$P' = \bigcup \{K : K \cap L' \neq \emptyset \wedge K \text{ is a generator of } W\}.$$

Then P' is a plane such that $P' \subseteq W$ and $P \cap P'$ is a line. But $P \cap P' \subseteq P \cap W$. We get a contradiction. $\square$

Theorem. All cylinders of revolution and elliptic cylinders are identical from the affine point of view.

Proof. Follows directly from the fact that all ellipses and circles are identical from the affine point of view. $\square$

Cone of revolution:

Definition.

A *cone of revolution* $\underset{df}{=}$ a set built by revolution of a line L about a line M under the assumption $\overline{\overline{L \cap M}} = 1$ and $\sim L \perp M$.

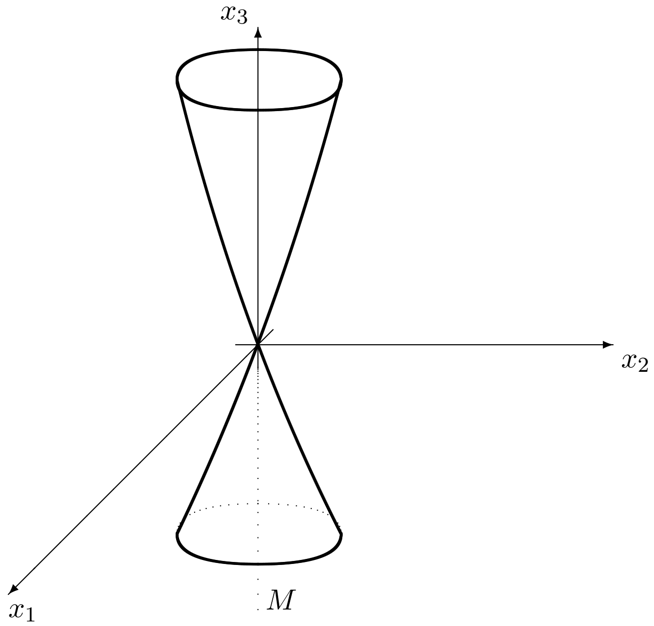
Let $\alpha \in \mathbb{R}$ and $M = L_3 = x_3$-axis. Take a line

$$L : \begin{cases} x_3 = \alpha x_2, \\ x_1 = 0, \end{cases}$$

that is,

$$L : \begin{cases} \varphi(x_2, x_3) = x_3 - \alpha x_2 = 0, \\ x_1 = 0 \end{cases}$$

and revolve it about M:



From theorem on an equation of a set of revolution we have

$$S = S(L, L_3) : \left(x_3 - \alpha\sqrt{x_1^2 + x_2^2}\right)\left(x_3 + \alpha\sqrt{x_1^2 + x_2^2}\right) = 0, \text{ that is,}$$
$$S : x_3^2 - \alpha^2\left(x_1^2 + x_2^2\right) = 0.$$

Hence

$$S : \alpha^2\left(x_1^2 + x_2^2\right) = x_3^2.$$

That is the *canonical equation of a cone of revolution.* Then $L \cap M$ is called a *vertex* of a cone, and any line passing through a vertex = a rectilinear generator of a cone. If $\alpha = 0$, then a cone reduces to the plane $x_3 = 0$. If $\alpha = 1$, then $S : x_1^2 + x_2^2 = x_3^2$ is the *unit cone.*

Theorem. A cone of revolution in canonical position is symmetric with respect to each of the coordinate planes and also with respect to each coordinate axes. Moreover a vertex of a cone is its centre of symmetry.

Proof. Follows from the form of a canonical equation of a cone. □

Take the unit cone $S : x_1^2 + x_2^2 = x_3^2$ and the affine transformation $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$f(x_1, x_2, x_3) = (\alpha_1 x_1, \alpha_2 x_2, x_3), \text{ where } \alpha_1, \alpha_2 > 0 \text{ and } \alpha_1 \neq \alpha_2.$$

Then f transforms S onto the set

$$SE : \frac{x_1^2}{\alpha_1^2} + \frac{x_2^2}{\alpha_2^2} = x_3^2.$$

That is the *canonical equation of an elliptic cone.*

Conclusion. All cones of revolution and elliptic cones are identical from the affine point of view.

Theorem. (**On generators of a cone**)

If S is a cone of revolution (or an elliptic cone), then through every $x \in S$ distinct from a vertex passes exactly one rectilinear generator of the cone S.

Proof. Obvious. □

Definition. (**Ruled set**)

A *ruled set* is a set which is a union of lines.

Theorem. (**On characterization of ruled sets**) $X \subseteq \mathbb{R}^3$

$$X \text{ is a ruled set } \Leftrightarrow \bigwedge_{x \in X} \bigvee_{L - \text{a line}} x \in L \subseteq X.$$

Proof.

($\Rightarrow$) $X = \bigcup_{t \in T} L_t$, $\{L_t : t \in T\}$ – a set of lines, $x \in X$

Then $x \in \bigcup_{t \in T} L_t$, whence $\bigvee_{t \in T} x \in L_t \subseteq X$.

($\Leftarrow$) $\bigwedge_{x \in X} \bigvee_{L - \text{a line}} x \in L \subseteq X$, that is,

$$\bigwedge_{x \in X} \bigvee_{L_x - \text{a line}} x \in L_x \subseteq X.$$

Take $\{L_x\}_{x \in X}$. Then $X = \bigcup_{x \in X} L_x$. Thus X is a ruled set. □

Theorem. The notion of a ruled set is an affine invariant.

Proof. $f : \mathbb{R}^3 \to \mathbb{R}^3$ – an affine transformation, $X = \bigcup_{t \in T} L_t$, $\{L_t : t \in T\}$ – a set of lines

Then

$$f(X) = f\left(\bigcup_{t \in T} L_t\right) = \bigcup_{t \in T} f(L_t)$$

and $\{f(L_t)\}_{t\in T}$ is a set of lines. Thus $f(X)$ is a ruled set. $\square$

Conclusion. All cylinders and cones are ruled sets.

Remark. Note that every conic can be obtained as a section of a cone of revolution by some plane. Therefore the parabolas, ellipses and hyperbolas have the common name of *conics*.

We also have other definitions of a cylinder of revolution and a cone of revolution:

1. $M \subseteq \mathbb{R}^3$ – a line, $r > 0$

A cylinder of revolution can also be defined in the following way:

$$W(M,r) \underset{df}{=} \{x \in \mathbb{R}^3 : \rho(x,M) = r\}.$$

2. $\mathfrak{a}, a \in \mathbb{R}^3$, $\mathfrak{a} \neq 0$, $0 < \beta < \frac{\pi}{2}$

A cone of revolution can also be defined in the following way:

$$S(a,\mathfrak{a},\beta) \underset{df}{=} \{x \in \mathbb{R}^3 : x = a \ \vee \ \sphericalangle(\mathfrak{a},[x-a]) = \beta \ \vee \ \sphericalangle(\mathfrak{a},[x-a]) = \pi - \beta\}.$$

Ellipsoid:

Definition.

An *ellipsoid of revolution* $\underset{df}{=}$ a set built by revolution of an ellipse about one of its axes of symmetry.

An ellipsoid of revolution is called *prolate* when the revolution is about the major axis of an ellipse, and *oblate* when the revolution is about the minor axis of an ellipse.

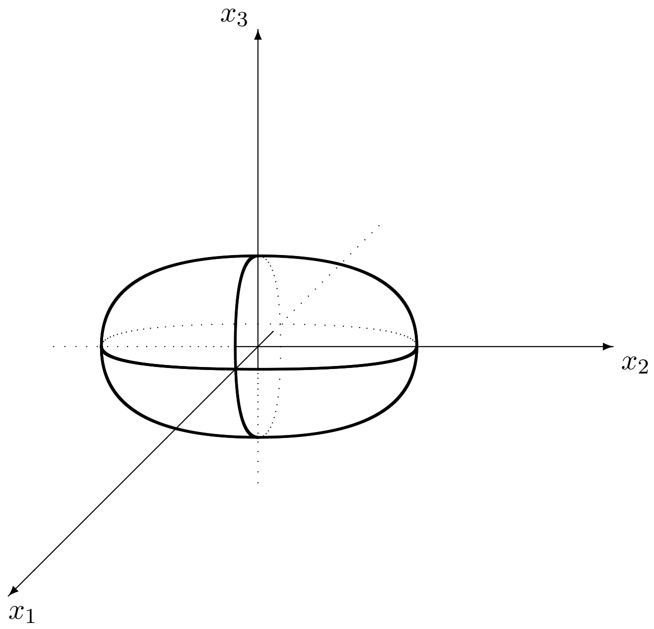
Let $L_3 = x_3$-axis. Take an ellipse

$$E : \begin{cases} \frac{x_2^2}{\alpha_2^2} + \frac{x_3^2}{\alpha_3^2} = 1, \\ x_1 = 0, \end{cases}$$

that is,

$$E : \begin{cases} \varphi(x_2,x_3) = \frac{x_2^2}{\alpha_2^2} + \frac{x_3^2}{\alpha_3^2} - 1 = 0, \\ x_1 = 0 \end{cases}$$

and revolve it about the axis L_3:



From theorem on an equation of a set of revolution we have

$$S(E, L_3) : \frac{x_1^2 + x_2^2}{\alpha_2^2} + \frac{x_3^2}{\alpha_3^2} = 1$$

That is the *canonical equation of an ellipsoid of revolution.* If $0 < \alpha_2 < \alpha_3$, then an ellipsoid of revolution is prolate, and if $0 < \alpha_3 < \alpha_2$, then an ellipsoid of revolution is oblate.

Take the affine transformation $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$f(x_1, x_2, x_3) = \left(\frac{\alpha_1}{\alpha_2} x_1, x_2, x_3\right), \text{ where } \alpha_1, \alpha_2 > 0 \text{ and } \alpha_1 \neq \alpha_2.$$

Then f transforms an ellipsoid of revolution onto the set

$$\frac{x_1^2}{\alpha_1^2} + \frac{x_2^2}{\alpha_2^2} + \frac{x_3^2}{\alpha_3^2} = 1.$$

That is the *canonical equation of a three-axis ellipsoid* (simply, an *ellipsoid*).

Moreover, the affine transformation $g : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$g(x_1, x_2, x_3) = \left(\frac{1}{\alpha_1} x_1, \frac{1}{\alpha_2} x_2, \frac{1}{\alpha_3} x_3\right)$$

transforms an ellipsoid into the sphere with the equation $x_1^2 + x_2^2 + x_3^2 = 1$.

Conclusion. All ellipsoids and spheres are identical from the affine point of view.

Theorem. An ellipsoid in canonical position is symmetric with respect to each coordinate plane and each coordinate axis, and with respect to the origin.

Proof. Follows from the form of the canonical equation of an ellipsoid. □

Remark. Points $(\alpha_1, 0, 0)$, $(-\alpha_1, 0, 0)$, $(0, \alpha_2, 0)$, $(0, -\alpha_2, 0)$, $(0, 0, \alpha_3)$ and $(0, 0, -\alpha_3)$ are called vertices of an ellipsoid with the equation

$$\frac{x_1^2}{\alpha_1^2} + \frac{x_2^2}{\alpha_2^2} + \frac{x_3^2}{\alpha_3^2} = 1.$$

Theorem. An ellipsoid of revolution (a three-axis ellipsoid) is not a ruled set.

Proof. Follows from the form of the canonical equation of an ellipsoid and theorem on characterization of ruled sets. □

Hyperboloid of one sheet:

Definition.

A *hyperboloid of revolution of one sheet* $\underset{df}{=}$ a set built by revolution of a hyperbola about an axis of symmetry which does not intersect a hyperbola.

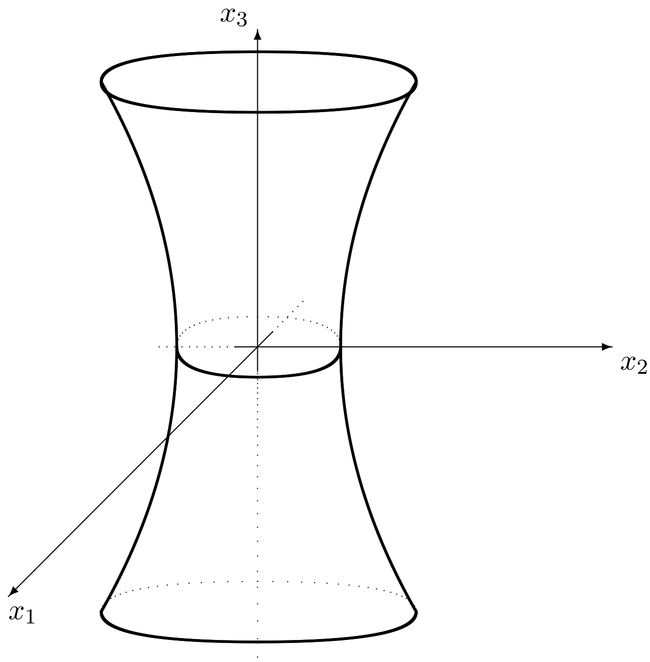
Let $L_3 = x_3$-axis. Take a hyperbola

$$H : \begin{cases} \frac{x_2^2}{\alpha_2^2} - \frac{x_3^2}{\alpha_3^2} = 1, \\ x_1 = 0, \end{cases}$$

that is,

$$H : \begin{cases} \varphi(x_2, x_3) = \frac{x_2^2}{\alpha_2^2} - \frac{x_3^2}{\alpha_3^2} - 1 = 0, \\ x_1 = 0 \end{cases}$$

and revolve it about the axis L_3:



From theorem on an equation of a set of revolution we have

$$S(H, L_3) : \frac{x_1^2 + x_2^2}{\alpha_2^2} - \frac{x_3^2}{\alpha_3^2} = 1$$

That is the *canonical equation of a hyperboloid of revolution of one sheet.*

Take the affine transformation $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$f(x_1, x_2, x_3) = \left(\frac{\alpha_1}{\alpha_2} x_1, x_2, x_3\right), \text{ where } \alpha_1, \alpha_2 > 0 \text{ and } \alpha_1 \neq \alpha_2.$$

Then f transforms a hyperboloid of revolution of one sheet onto the set

$$H_1 : \frac{x_1^2}{\alpha_1^2} + \frac{x_2^2}{\alpha_2^2} - \frac{x_3^2}{\alpha_3^2} = 1.$$

That is the *canonical equation of a hyperboloid of one sheet.*

Moreover, the affine transformation $g : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$g(x_1, x_2, x_3) = \left(\frac{1}{\alpha_1} x_1, \frac{1}{\alpha_2} x_2, \frac{1}{\alpha_3} x_3\right)$$

transforms a hyperboloid of one sheet into the hyperboloid of one sheet with the equation $x_1^2 + x_2^2 - x_3^2 = 1$.

Conclusion. All hyperboloids of one sheet are identical from the affine point of view.

Theorem. A hyperboloid of one sheet in canonical position is symmetric with respect to each coordinate plane and each coordinate axis, and with respect to the origin.

Proof. Follows from the form of the canonical equation of a hyperboloid of one sheet. □

Theorem. $F \subseteq \mathbb{R}^3$ – an algebraic set

F is a hyperboloid of revolution of one sheet iff it is a set built by revolution of a line L about a line M such that $L \cap M = \emptyset$ and $\sim L \perp M$.

Proof. Let $M = L_3 = x_3$-axis and

$$L : \begin{cases} x_2 = a, \\ x_1 = bx_3, \ b \neq 0. \end{cases}$$

Then a set built by revolution of L about M is a union of circles lying on planes $x_3 = t$, with centres on M and intersecting L, that is, the set:

$$F = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : \bigvee_{t \in \mathbb{R}} x_3 = t \ \wedge \ x_1^2 + x_2^2 = a^2 + (bt)^2 \right\}.$$

Hence

$$F : x_1^2 + x_2^2 - b^2 x_3^2 = a^2.$$

Setting $a = \alpha_2$ and $b = \frac{\alpha_2}{\alpha_3}$ we get a canonical equation of a hyperboloid of revolution of one sheet. □

Remark. The same hyperboloid of revolution of one sheet can be obtained by taking the line

$$L' : \begin{cases} x_2 = a, \\ x_1 = -bx_3, \ b \neq 0 \end{cases}$$

instead of L.

Conclusion. Through every point of a hyperboloid of one sheet there pass two lines which lie entirely on it.

Conclusion. A hyperboloid of one sheet is a ruled set.

Hyperboloid of two sheets:

Definition.

A *hyperboloid of revolution of two sheets* $\underset{df}{=}$ a set built by revolution of a hyperbola about its axis of symmetry which intersects a hyperbola.

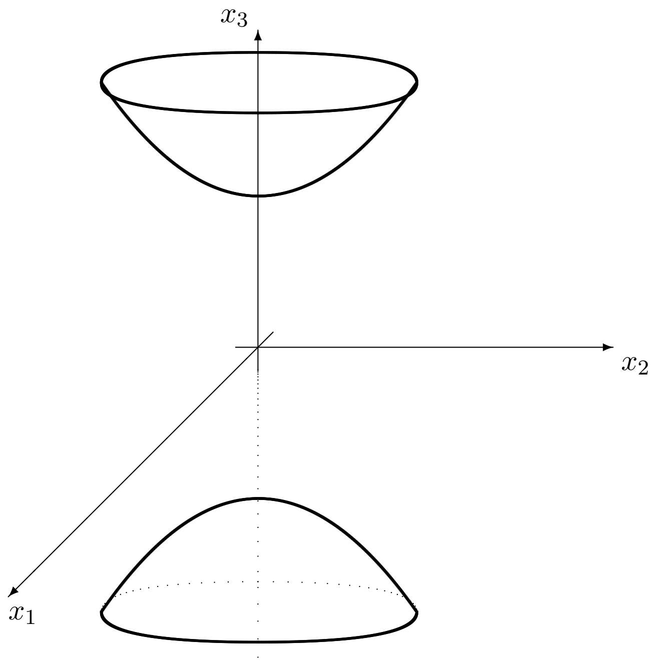
Let $L_3 = x_3$-axis. Take a hyperbola

$$H : \begin{cases} -\frac{x_2^2}{\alpha_2^2} + \frac{x_3^2}{\alpha_3^2} = 1, \\ x_1 = 0, \end{cases}$$

that is,

$$H : \begin{cases} \varphi(x_2, x_3) = -\frac{x_2^2}{\alpha_2^2} + \frac{x_3^2}{\alpha_3^2} - 1 = 0, \\ x_1 = 0 \end{cases}$$

and revolve it about the axis L_3:



From theorem on an equation of a set of revolution we have

$$S(H, L_3) : \frac{x_1^2 + x_2^2}{\alpha_2^2} - \frac{x_3^2}{\alpha_3^2} = -1$$

That is the *canonical equation of a hyperboloid of revolution of two sheets.*

Take the affine transformation $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$f(x_1, x_2, x_3) = \left(\frac{\alpha_1}{\alpha_2} x_1, x_2, x_3 \right), \text{ where } \alpha_1 > 0.$$

Then f transforms a hyperboloid of revolution of two sheets onto the set

$$H_2 : \frac{x_1^2}{\alpha_1^2} + \frac{x_2^2}{\alpha_2^2} - \frac{x_3^2}{\alpha_3^2} = -1.$$

That is the *canonical equation of a hyperboloid of two sheets.*

Moreover, the affine transformation $g : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$g(x_1, x_2, x_3) = \left(\frac{1}{\alpha_1} x_1, \frac{1}{\alpha_2} x_2, \frac{1}{\alpha_3} x_3 \right)$$

transforms it into the hyperboloid of two sheets with the equation $x_1^2 + x_2^2 - x_3^2 = -1$.

Conclusion. All hyperboloids of two sheets are identical from the affine point of view.

Theorem. A hyperboloid of two sheets in canonical position is symmetric with respect to each coordinate plane and each coordinate axis, and with respect to the origin.

Proof. Follows from the form of the canonical equation of a hyperboloid of two sheets. □

Theorem. A hyperboloid of two sheets is not a ruled set.

Proof. Since the notion of a ruled set is an affine invariant, it suffices to show that a hyperboloid of revolution of two sheets $H_2 : x_1^2 + x_2^2 - x_3^2 = -1$ is not a ruled set. We show that through point $a = (0, 0, 1)$ does not pass any generator L of the hyperboloid H_2.

Let $L = \{a + t\mathfrak{a} : t \in \mathbb{R}\}$, where $\mathfrak{a} = [\alpha_1, \alpha_2, \alpha_3] \neq [0, 0, 0]$. Suppose that $L \subseteq H_2$. Then

$$\bigwedge_{t\in\mathbb{R}} (t\alpha_1)^2 + (t\alpha_2)^2 - (1 + t\alpha_3)^2 = -1,$$

that is,

$$\bigwedge_{t\in\mathbb{R}} t^2(\alpha_1^2 + \alpha_2^2 - \alpha_3^2) - 2t\alpha_3 = 0.$$

Hence $\alpha_1^2 + \alpha_2^2 - \alpha_3^2 = 0$ and $\alpha_3 = 0$, that is, $\mathfrak{a} = [0, 0, 0]$. We get a contradiction. □

Elliptic paraboloid:

Definition.

A *paraboloid of revolution* $\underset{df}{=}$ a set built by revolution of a parabola about its axis of symmetry.

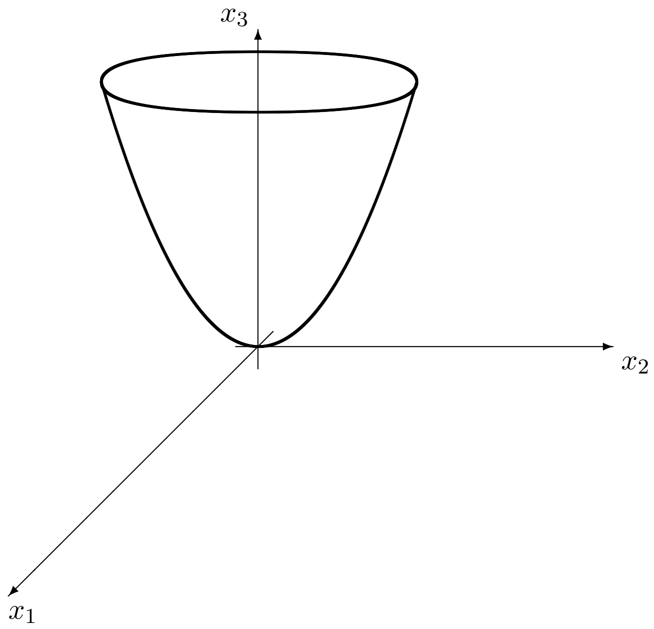
Let $L_3 = x_3$-axis. Take a parabola

$$P : \begin{cases} \frac{x_2^2}{\alpha_2^2} = 2x_3, \\ x_1 = 0, \end{cases}$$

that is,

$$P : \begin{cases} \varphi(x_2, x_3) = \frac{x_2^2}{\alpha_2^2} - 2x_3 = 0, \\ x_1 = 0 \end{cases}$$

and revolve it about the axis L_3:



From theorem on an equation of a set of revolution we have

$$S(P, L_3) : \frac{x_1^2 + x_2^2}{\alpha_2^2} = 2x_3.$$

That is the *canonical equation of a paraboloid of revolution.*

Take the affine transformation $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$f(x_1, x_2, x_3) = \left(\frac{\alpha_1}{\alpha_2} x_1, x_2, x_3\right), \text{ where } \alpha_1 > 0.$$

Then f transforms a paraboloid of revolution onto the set

$$PE : \frac{x_1^2}{\alpha_1^2} + \frac{x_2^2}{\alpha_2^2} = 2x_3.$$

That is the *canonical equation of an elliptic paraboloid.*

Moreover, the affine transformation $g : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$g(x_1, x_2, x_3) = \left(\frac{1}{\alpha_1} x_1, \frac{1}{\alpha_2} x_2, x_3\right)$$

transforms it into the paraboloid of revolution with the equation $x_1^2 + x_2^2 = 2x_3$.

Conclusion. All paraboloids of revolution and elliptic paraboloids are identical from the affine point of view.

Theorem. A paraboloid of revolution (an elliptic paraboloid) in canonical position is symmetric with respect to the planes $x_1 = 0$ and $x_2 = 0$, and with respect to the x_3-axis.

Proof. Follows from the form of the canonical equation of a paraboloid. □

Remark. The intersecting point of a paraboloid with its axis of symmetry is called the vertex of a paraboloid.

Theorem. A paraboloid of revolution (an elliptic paraboloid) does not have a centre of symmetry.

Proof. In fact, such centre could not be different from the vertex, since the point symmetric to the vertex would also have to be a vertex. But the vertex is not a centre of symmetry, since points $(0, \alpha_2, \frac{1}{2})$ and $(0, -\alpha_2, -\frac{1}{2})$ are symmetric with respect to the vertex $(0, 0, 0)$ of a paraboloid in canonical position. The first of them lies on the paraboloid, while the other one does not. □

Theorem. A paraboloid of revolution (an elliptic paraboloid) is not a ruled set.

Proof. Similar to that of a hyperboloid of two sheets. □

Hyperbolic paraboloid:

$Q_1, Q_2 \subseteq \mathbb{R}^3$ – planes, $Q_1 \perp Q_2$, $P_1, P_2 \subseteq \mathbb{R}^3$ – parabolas, $P_1 \subseteq Q_1$, $P_2 \subseteq Q_2$

$a \in \mathbb{R}^3$ – a common vertex of parabolas P_1 and P_2

$L \subseteq \mathbb{R}^3$ – a common axis of symmetry of parabolas P_1 and P_2

$b \in P_2$, $f_b : \mathbb{R}^3 \to \mathbb{R}^3$ – a translation given by

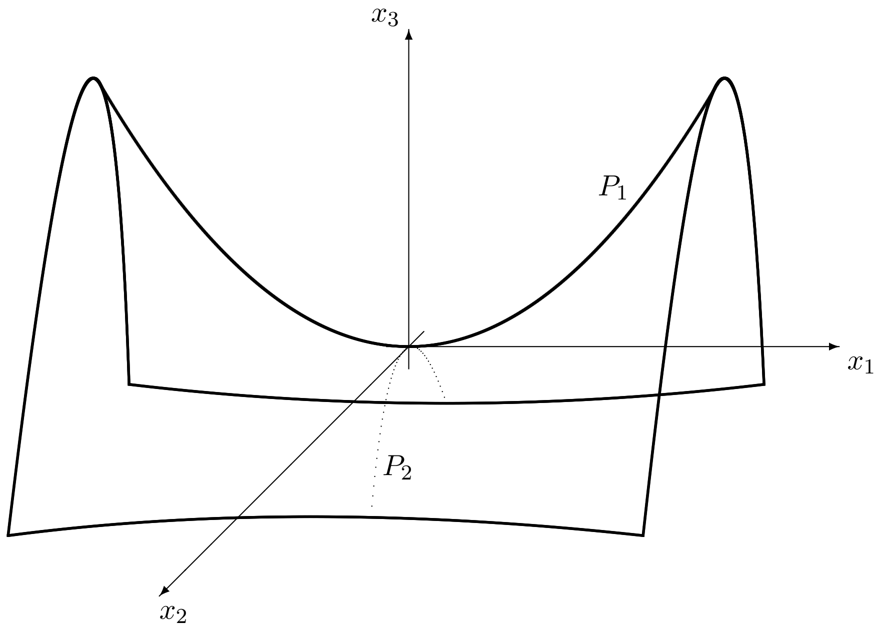
$$f_b(x) = x + (b - a).$$

We have: $f_b(a) = b$ and $f_b(P_1)$ is a parabola.

Definition.

A *hyperbolic paraboloid*:

$$PH \underset{df}{=} \bigcup_{b \in P_2} f_b(P_1).$$



Let $Q_1 : x_2 = 0,\ \ Q_2 : x_1 = 0,\ \ a = (0,0,0),\ \ L = L_3 = x_3$-axis.

Then

$$P_1 : \begin{cases} x_1^2 - 2\alpha_1^2 x_3 = 0, \\ x_2 = 0 \end{cases} \quad \text{and} \quad P_2 : \begin{cases} x_2^2 + 2\alpha_2^2 x_3 = 0, \\ x_1 = 0, \end{cases}$$

that is,

$$P_1 : \begin{cases} x_3 = \frac{x_1^2}{2\alpha_1^2}, \\ x_2 = 0 \end{cases} \quad \text{and} \quad P_2 : \begin{cases} x_3 = -\frac{x_2^2}{2\alpha_2^2}, \\ x_1 = 0 \end{cases}$$

and $f_b(x) = x + b$, where $b \in P_2$.

Hence

$$PH = \bigcup_{b \in P_2} f_b(P_1) = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = \frac{x_1^2}{2\alpha_1^2} - \frac{x_2^2}{2\alpha_2^2} \right\}.$$

Thus

$$PH : \frac{x_1^2}{\alpha_1^2} - \frac{x_2^2}{\alpha_2^2} = 2x_3.$$

That is the *canonical equation of a hyperbolic paraboloid.*

Remark. Another name of a hyperbolic paraboloid is a *saddle surface* or simply a *saddle.*

Take the affine transformation $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$f(x_1, x_2, x_3) = \left(\frac{1}{\alpha_1} x_1, \frac{1}{\alpha_2} x_2, x_3 \right).$$

Then f transforms a hyperbolic paraboloid onto a hyperbolic paraboloid with the equation $x_1^2 - x_2^2 = 2x_3$.

Conclusion. All hyperbolic paraboloids are identical from the affine point of view.

Theorem. A hyperbolic paraboloid in canonical position is symmetric with respect to the planes $x_1 = 0$ and $x_2 = 0$, and with respect to the x_3-axis.

Proof. Follows from the form of the canonical equation of a paraboloid. □

Remark. The point $(0, 0, 0)$ is called the vertex of a hyperbolic paraboloid in canonical position.

Theorem. A hyperbolic paraboloid does not have a centre of symmetry.

Proof. Similar to that of an elliptic paraboloid. □

Theorem. A hyperbolic paraboloid is a ruled set.

Proof. It suffices to show that the hyperbolic paraboloid $PH : x_1^2 - x_2^2 = 2x_3$ is a ruled set. Remark that

$$PH : \begin{vmatrix} x_1 - x_2 & x_3 \\ 2 & x_1 + x_2 \end{vmatrix} = 0.$$

Hence

$$(x_1, x_2, x_3) \in PH \;\Leftrightarrow \bigvee_{\alpha,\beta\in\mathbb{R},\alpha^2+\beta^2>0} \begin{cases} \alpha(x_1 - x_2) + \beta x_3 = 0, \\ 2\alpha + \beta(x_1 + x_2) = 0 \end{cases} \text{(proportional columns)}$$

and

$$(x_1, x_2, x_3) \in PH \;\Leftrightarrow \bigvee_{\gamma,\delta\in\mathbb{R},\gamma^2+\delta^2>0} \begin{cases} \gamma(x_1 - x_2) + 2\delta = 0, \\ \gamma x_3 + \delta(x_1 + x_2) = 0 \end{cases} \text{(proportional rows).}$$

The first system is the edge equation of some line $L_{\alpha\beta}$, since $\mathfrak{a}_1 = [\alpha, -\alpha, \beta] \perp L_{\alpha\beta}$, $\mathfrak{a}_2 = [\beta, \beta, 0] \perp L_{\alpha\beta}$ and $\mathfrak{a}_1 \nparallel \mathfrak{a}_2$ (since $(\alpha, \beta) \neq (0, 0)$). Similarly, the second system describes some line $L_{\gamma\delta}$. Thus

$$x \in PH \;\Leftrightarrow \bigvee_{\alpha,\beta\in\mathbb{R},\alpha^2+\beta^2>0} x \in L_{\alpha\beta} \;\Leftrightarrow \bigvee_{\gamma,\delta\in\mathbb{R},\gamma^2+\delta^2>0} x \in L_{\gamma\delta},$$

whence

$$PH = \bigcup_{\alpha,\beta\in\mathbb{R},\alpha^2+\beta^2>0} L_{\alpha\beta} \quad \text{and} \quad PH = \bigcup_{\gamma,\delta\in\mathbb{R},\gamma^2+\delta^2>0} L_{\gamma\delta}.$$

Thus a hyperbolic paraboloid is a ruled set. □

Conclusion. Through every point of a hyperbolic paraboloid there pass two lines which lie entirely on it.

Remark. Ellipsoids, hyperboloids of one and two sheets and elliptic and hyperbolic paraboloids together are called *quadrics*.

Conclusion. Quadrics are algebraic sets of the second degree in $\mathbb{R}^3$.

Remark. Any two quadrics are not identical from the affine point of view, that is, they represent different affine types.

9. Projective spaces: real P^n and complex CP^n

Definition. (**Homogeneous coordinates**) $\lambda \in \mathbb{R}\backslash\{0\}, \;\; x = (x_1, \dots, x_n) \in \mathbb{R}^n$

Homogeneous coordinates of a point $x \underset{df}{=} \{\lambda, \lambda x_1, \dots, \lambda x_n\}$.

Denotation: $\{x_0, x_1, \dots, x_n\}$.

Hence for $x_0 \neq 0$ we have

$$\{x_0, x_1, \dots, x_n\} = \left\{1, \frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right\} = \left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) \in \mathbb{R}^n.$$

If $x_0 \to 0$, then the distance of points $\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)$ and $(x_1, \dots, x_n)$ increases to infinity. Thus a point $\{0, x_1, \dots, x_n\}$ is called a *point at infinity*. It is easy to see, that

$$\{0, x_1, \dots, x_n\} = \mathcal{K}([x_1, \dots, x_n]),$$

that is, a point at infinity $\{0, x_1, \dots, x_n\}$ is a direction of a vector $[x_1, \dots, x_n]$ in $\mathbb{R}^n$.

Definition. (**n-dimensional projective space P^n**)

$$P^n \underset{df}{=} \mathbb{R}^n \cup \{\text{directions in } \mathbb{R}^n\}.$$

Directions in $\mathbb{R}^n$ are called *improper points* of a projective space P^n. The space P^1 is called a *projective line*, and the space P^2 is called a *projective plane*.

Definition. (**Projective line in P^n**)

In P^1 there is exactly one projective line. It is P^1.

If projective lines have already been defined in the space P^{n-1}, then in the space P^n projective lines are:

1) lines in $\mathbb{R}^n$ together with their improper points (*proper lines*),

2) sets of points of the form $\{0, x_1, \dots, x_n\}$ such that the set of points $\{x_1, \dots, x_n\} \in P^{n-1}$ forms a projective line in P^{n-1} (*improper lines*).

Remark. The set of improper points of the projective plane P^2 is an improper line.

Remark. A projective line differs from a Cartesian line by an additional point which, in a sense, closes it, making it similar to a circle with an "infinitely large" radius.

Theorem. Through every two different points $a = \{a_0, a_1, \dots, a_n\}, b = \{b_0, b_1, \dots, b_n\} \in P^n$ there passes exactly one projective line consisted of points

$$x(\lambda, \mu) = \{\lambda a_0 + \mu b_0, \lambda a_1 + \mu b_1, \dots, \lambda a_n + \mu b_n\},$$

where $(\lambda,\mu)\in\mathbb{R}^2\backslash\{(0,0)\}$. That is the *parametric equation* of a projective line.

Proof. We have $a\neq b$ and $(\lambda,\mu)\neq(0,0)$, whence $x(\lambda,\mu)\in P^n$. Moreover, for $\alpha,\beta\neq 0$

$$\begin{aligned} x(\lambda,\mu) &= \{\lambda a_0+\mu b_0, \lambda a_1+\mu b_1,\dots,\lambda a_n+\mu b_n\}\\ &= \{\lambda\alpha a_0+\mu\beta b_0, \lambda\alpha a_1+\mu\beta b_1,\dots,\lambda\alpha a_n+\mu\beta b_n\}, \end{aligned}$$

that is, any point proportional to a and any point proportional to b determine the same points $x(\lambda,\mu)$.

We prove by induction with respect to n that points of above form determine a projective line. For $n=1$ it is obvious.

Assume that theorem is true in projective spaces of dimensions lower than n. We have three cases:

1) a,b – proper points

Assume that $a_0=b_0=1$. If $\lambda+\mu\neq 0$, then

$$\begin{aligned} x(\lambda,\mu) &= \left(\frac{\lambda}{\lambda+\mu}a_1+\frac{\mu}{\lambda+\mu}b_1,\dots,\frac{\lambda}{\lambda+\mu}a_n+\frac{\mu}{\lambda+\mu}b_n\right)\\ &= \frac{\lambda+\mu-\mu}{\lambda+\mu}a+\frac{\mu}{\lambda+\mu}b=\left(1-\frac{\mu}{\lambda+\mu}\right)a+\frac{\mu}{\lambda+\mu}b. \end{aligned}$$

By theorem on a line a point $x(\lambda,\mu)$ is a proper point of the line. If $\lambda+\mu=0$, then

$$\begin{aligned} x(\lambda,\mu) &= \{\lambda a_0+\mu b_0, \lambda a_1+\mu b_1,\dots,\lambda a_n+\mu b_n\}\\ &\overset{\lambda=-\mu}{=} \{0, b_1-a_1,\dots,b_n-a_n\} \end{aligned}$$

is an improper point of the line.

2) a – proper, b – improper (or vice versa)

Assume that $a_0=1$ and $b_0=0$. If $\lambda\neq 0$, then

$$x(\lambda,\mu)=\left(a_1+\frac{\mu}{\lambda}b_1,\dots,a_n+\frac{\mu}{\lambda}b_n\right)=a+\frac{\mu}{\lambda}(b_1,\dots,b_n).$$

So we see that a point $x(\lambda,\mu)$ is a proper point of the line that passes through a and which has a direction $[b_1,\dots,b_n]$, that is, an improper point $\{0,b_1,\dots,b_n\}$. If $\lambda=0$, then

$$x(0,\mu)=\{0,\mu b_1,\dots,\mu b_n\}=b,$$

that is, it is an improper point of the line.

3) a,b – improper points

Then $a_0=b_0=0$ and

$$x(\lambda,\mu)=\{0,\lambda a_1+\mu b_1,\dots,\lambda a_n+\mu b_n\}.$$

From assumption, $\{\lambda a_1 + \mu b_1, \ldots, \lambda a_n + \mu b_n\}$ presents a projective line in P^{n-1}. Hence $x(\lambda, \mu)$ presents a projective line in P^n.

It is easy to show that numbers λ and μ are determined by the point $x(\lambda, \mu)$ up to a constant of proportionality. $\square$

Definition. $f : P^n \to P^n$, $\{x_0, x_1, \ldots, x_n\} \in P^n$

f is a *projective transformation* $\underset{df}{\Leftrightarrow}$ $f(x_0, x_1, \ldots, x_n) = \{\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_n\}$, where $\overline{x}_j = \alpha_{0j}x_0 + \alpha_{1j}x_1 + \ldots + \alpha_{nj}x_n$ for $j = 0, 1, \ldots, n$ and a matrix of f:

$$A_f = \begin{bmatrix} \alpha_{00} & \alpha_{10} & \ldots & \alpha_{n0} \\ \alpha_{01} & \alpha_{11} & \ldots & \alpha_{n1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{0n} & \alpha_{1n} & \ldots & \alpha_{nn} \end{bmatrix}$$

is nonsingular.

Conclusion. A projective transformation is a one-to-one transformation.

Theorem. Composition of two projective transformations is a projective transformation.

Proof. $f : P^n \to P^n$ – a projective transformation with a matrix A_f, $f' : P^n \to P^n$ – a projective transformation with a matrix $A_{f'}$

Hence f has the form

$$\begin{bmatrix} \overline{x}_0 \\ \overline{x}_1 \\ \vdots \\ \overline{x}_n \end{bmatrix} = A_f \cdot \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{bmatrix}$$

and f' has the form

$$\begin{bmatrix} x'_0 \\ x'_1 \\ \vdots \\ x'_n \end{bmatrix} = A_{f'} \cdot \begin{bmatrix} \overline{x}_0 \\ \overline{x}_1 \\ \vdots \\ \overline{x}_n \end{bmatrix}.$$

Then a transformation $f'f$ can be written as

$$\begin{bmatrix} x'_0 \\ x'_1 \\ \vdots \\ x'_n \end{bmatrix} = A_{f'} \cdot A_f \cdot \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Thus $A_{f'f} = A_{f'} \cdot A_f$ and it is nonsingular, since A_f and $A_{f'}$ are nonsingular. Hence a transformation $f'f$ is projective. $\square$

Theorem. If f is a projective transformation, then f^{-1} is a projective transformation.

Proof. $f : P^n \to P^n$ – a projective transformation with a matrix A_f

So f has the form

$$\begin{bmatrix} \overline{x}_0 \\ \overline{x}_1 \\ \vdots \\ \overline{x}_n \end{bmatrix} = A_f \cdot \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

That is the system of linear equations with nonsingular coefficient matrix A_f. So it has precisely one solution $x_0, x_1, \ldots, x_n$. Solving that system we get the transformation f^{-1} of the form

$$\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{bmatrix} = A_f^{-1} \cdot \begin{bmatrix} \overline{x}_0 \\ \overline{x}_1 \\ \vdots \\ \overline{x}_n \end{bmatrix}.$$

Hence $A_{f^{-1}} = A_f^{-1}$ and it is nonsingular. Thus f^{-1} is a projective transformation. $\square$

Definition.

A *projective invariant* $\underset{df}{=}$ a property which is unchanged by projective transformations.

Theorem. A projective line is a projective invariant.

Proof. $a = \{a_0, a_1, \ldots, a_n\}, b = \{b_0, b_1, \ldots, b_n\} \in P^n$, $a \neq b$

Let L be a projective line which passes through points a, b. Then

$$L : x(\lambda, \mu) = \{\lambda a_0 + \mu b_0, \lambda a_1 + \mu b_1, \ldots, \lambda a_n + \mu b_n\}, \text{ where } (\lambda, \mu) \neq (0, 0).$$

It is easy to see that a point $x(\lambda, \mu)$ of L is transformed by a projective transformation into a point

$$\{\lambda \overline{a}_0 + \mu \overline{b}_0, \lambda \overline{a}_1 + \mu \overline{b}_1, \ldots, \lambda \overline{a}_n + \mu \overline{b}_n\},$$

that is, into a point of a projective line which passes through points $\overline{a} = \{\overline{a}_0, \overline{a}_1, \ldots, \overline{a}_n\}$ and $\overline{b} = \{\overline{b}_0, \overline{b}_1, \ldots, \overline{b}_n\}$. $\square$

Definition. $f : P^n \to P^n$ – a projective transformation, $\{x_0, x_1, \ldots, x_n\}, \{\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_n\} \in P^n$

A transformation f is an *affine transformation* $\underset{df}{\Leftrightarrow}$ $f(x_0, x_1, \ldots, x_n) = \{\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_n\}$, where

$$\begin{cases} \overline{x}_0 = x_0, \\ \overline{x}_j = \alpha_{0j} x_0 + \alpha_{1j} x_1 + \ldots + \alpha_{nj} x_n \quad \text{for} \quad j = 0, 1, \ldots, n. \end{cases}$$

Conclusion. Under projective affine transformations proper points of P^n go into proper ones, and improper points into improper ones.

Remark. A matrix of a projective affine transformation f has the form:

$$A_f = \begin{bmatrix} 1 & 0 & \dots & 0 \\ \alpha_{01} & \alpha_{11} & \dots & \alpha_{n1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{0n} & \alpha_{1n} & \dots & \alpha_{nn} \end{bmatrix}$$

and it is nonsingular. If A_f is orthogonal, then f is called a projective isometry, and if for $\lambda > 0$ a matrix $\frac{1}{\lambda}A_f$ is orthogonal, then f is called a projective similarity with the ratio λ.

Conclusion. Any affine transformation (isometry, similarity) is a projective transformation.

Conclusion. Any projective invariant is an affine invariant (so also a similarity invariant and an invariant of isometry).

Definition. (**Anharmonic ratio**)

L – a projective line in P^n, $p = \{p_0, p_1, \dots, p_n\}, q = \{q_0, q_1, \dots, q_n\} \in L$, $p \neq q$

$L : x(\lambda, \mu) = \{\lambda p_0 + \mu q_0, \lambda p_1 + \mu q_1, \dots, \lambda p_n + \mu q_n\}$

$a, b, c, d \in L$, $\overline{\overline{\{a, b, c, d\}}} = 4$, $a = x(\lambda_a, \mu_a)$, $b = x(\lambda_b, \mu_b)$, $c = x(\lambda_c, \mu_c)$, $d = x(\lambda_d, \mu_d)$

An *anharmonic ratio of points* a, b, c, d is given by

$$(a, b; c, d) = \frac{\begin{vmatrix} \lambda_a & \mu_a \\ \lambda_c & \mu_c \end{vmatrix} \cdot \begin{vmatrix} \lambda_b & \mu_b \\ \lambda_d & \mu_d \end{vmatrix}}{\begin{vmatrix} \lambda_a & \mu_a \\ \lambda_d & \mu_d \end{vmatrix} \cdot \begin{vmatrix} \lambda_b & \mu_b \\ \lambda_c & \mu_c \end{vmatrix}}, \quad (a, b; c, d) \neq 0.$$

If $a, b, c, d \in \mathbb{R}^n$, then

$$(a, b; c, d) = \pm \frac{\rho(a, c) \cdot \rho(b, d)}{\rho(a, d) \cdot \rho(b, c)}.$$

Theorem. L – a projective line in P^n, $a, b, c, d \in L$, $\overline{\overline{\{a, b, c, d\}}} = 4$

Then

1) $(a, b; c, d) = \frac{1}{(a,b;d,c)} = \frac{1}{(b,a;c,d)} = (b, a; d, c)$,
2) $(a, b; c, d) = (c, d; a, b)$,
3) $(a, b; c, d) = 1 - (a, c; b, d)$.

Proof. Follows directly from definition. □

Theorem. An anharmonic ratio is a projective invariant.

Proof. L – a projective line in P^n, $p = \{p_0, p_1, \dots, p_n\}, q = \{q_0, q_1, \dots, q_n\} \in L$, $p \neq q$

$L : x(\lambda, \mu) = \{\lambda p_0 + \mu q_0, \dots, \lambda p_n + \mu q_n\}$, $a, b, c, d \in L$, $\overline{\overline{\{a, b, c, d\}}} = 4$

Hence $a = x(\lambda_a, \mu_a)$, $b = x(\lambda_b, \mu_b)$, $c = x(\lambda_c, \mu_c)$, $d = x(\lambda_d, \mu_d)$.

Let $f : P^n \to P^n$ be a projective transformation such that $f(x_0, x_1, \dots, x_n) = \{\overline{x}_0, \overline{x}_1, \dots, \overline{x}_n\}$, where

$$\overline{x}_j = \alpha_{0j}x_0 + \alpha_{1j}x_1 + \ldots + \alpha_{nj}x_n, \quad j = 0, 1, \dots, n.$$

It is easy to see that f transforms points a, b, c, d respectively into points

$$\begin{aligned}
\overline{a} &= \{\lambda_a\overline{p}_0 + \mu_a\overline{q}_0, \dots, \lambda_a\overline{p}_n + \mu_a\overline{q}_n\}\\
\overline{b} &= \{\lambda_b\overline{p}_0 + \mu_b\overline{q}_0, \dots, \lambda_b\overline{p}_n + \mu_b\overline{q}_n\}\\
\overline{c} &= \{\lambda_c\overline{p}_0 + \mu_c\overline{q}_0, \dots, \lambda_c\overline{p}_n + \mu_c\overline{q}_n\}\\
\overline{d} &= \{\lambda_d\overline{p}_0 + \mu_d\overline{q}_0, \dots, \lambda_d\overline{p}_n + \mu_d\overline{q}_n\}
\end{aligned}$$

Thus $(a, b; c, d) = (\overline{a}, \overline{b}; \overline{c}, \overline{d})$ and proof is finished. □

Definition. L – a projective line in P^n, $a, b, c, d \in L$

A quadruple of points a, b, c, d is called *harmonic* $\underset{df}{\Leftrightarrow}$ $(a, b; c, d) = -1$.

Then a point d is called the *fourth harmonic of points* a, b, c. We can also say that pairs a, b and c, d are *harmonic conjugated.*

Example. If $a, b \in \mathbb{R}^n$, $a \neq b$, $c = \frac{a+b}{2}$ and $p_\infty \in L(a, b) \cap (P^n \backslash \mathbb{R}^n)$, then pairs a, b and c, p_∞ are harmonic conjugated. Indeed, we have $a = \{1, a_1, \dots, a_n\}$, $b = \{1, b_1, \dots, b_n\}$, $c = \{1, \frac{a_1+b_1}{2}, \dots, \frac{a_n+b_n}{2}\}$ and $p_\infty = \{0, b_1 - a_1, \dots, b_n - a_n\}$. Hence

$$(a, b; c, p_\infty) = \frac{\begin{vmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} \cdot \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} \cdot \begin{vmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix}} = \frac{\frac{1}{2} \cdot 1}{1 \cdot (-\frac{1}{2})} = -1.$$

Theorem. If a quadruple $(a, b; c, d)$ is harmonic, then also quadruples $(a, b; d, c)$, $(b, a; c, d)$, $(b, a; d, c)$ and $(c, d; a, b)$ are harmonic.

Proof. Follows from properties of an anharmonic ratio. □

Theorem. Harmonic quadruple and fourth harmonic are projective invariants.

Proof. Follows from the fact that an anharmonic ratio is a projective invariant. □

Definition. (**Projective plane in P^n**)

In P^2 there exists exactly one projective plane. It is P^2.

If projective planes have already been defined in the space P^{n-1}, then in the space P^n projective planes are:

1) planes in $\mathbb{R}^n$ together with their improper points of lines, which lie onto these planes (*proper planes*),

2) sets of points of the form $\{0, x_1, \ldots, x_n\}$ such that the set of points $\{x_1, \ldots, x_n\} \in P^{n-1}$ forms a projective plane in P^{n-1} (*improper planes*).

Remark. The set of improper points of a proper plane in P^n is an improper line.

Remark. Similarly, we define a k-dimensional projective hyperplane in P^n.

Remark. The set of improper points of the space P^n is an $(n-1)$-dimensional projective improper hyperplane. Particularly, the set of improper points of P^3 is an improper plane.

Theorem. Any two distinct lines in P^2 have exactly one common point (proper or improper).

Proof. Follows from definition of a projective line. $\square$

Theorem. Any two distinct planes in P^3 have exactly one common line (proper or improper).

Proof. Follows from definition of a projective plane. $\square$

Definition. (**Homogeneous polynomial**)

$\varphi : \mathbb{R}^n \to \mathbb{R}$ – a polynomial in n variables, $\varphi(x) = \sum\limits_{i_1,\ldots,i_n} \alpha_{i_1\ldots i_n} x_1^{i_1} \cdot \ldots \cdot x_n^{i_n}$, where

$x = (x_1, \ldots, x_n) \in \mathbb{R}^n,\ \ i_1, \ldots, i_n \in \{0, \ldots, k\},\ \ k \in \mathbb{N} \cup \{0\},\ \ \deg(\varphi) = k$

φ is *homogeneous* $\underset{df}{\Leftrightarrow}$ $(\alpha_{i_1\ldots i_n} \neq 0 \ \Rightarrow\ i_1 + \ldots + i_n = k)$.

Example.
1. $\varphi(x) = 2x_1x_2x_3 + x_1x_2^2 - x_3^3$ is the homogeneous polynomial of degree 3 in 3 variables.
2. $\varphi(x) = 2x_1^2 + x_2^2 + x_1x_2 - x_1$ is the nonhomogeneous polynomial.

Definition. $\varphi : \mathbb{R}^n \to \mathbb{R}$ – a homogeneous polynomial of degree k

An equation $\varphi(x) = 0$ is called the *homogeneous equation* of degree k.

Definition. (**An algebraic set in P^n**)

$\varphi : \mathbb{R}^n \to \mathbb{R}$ – a homogeneous polynomial of degree k, $\ F \subseteq P^n$

An *algebraic set in* P^n of degree k is the set:

$$F \underset{df}{=} \{\{x_0, x_1, \ldots, x_n\} \in P^n : \varphi(x_0, x_1, \ldots, x_n) = 0\}.$$

We will write $F : \varphi(x) = 0$.

Remarks. Similarly as in $\mathbb{R}^n$:
1. Algebraic sets of degree 0 in P^n: $\emptyset$ and P^n.
2. Algebraic sets of degree 1 in P^n: $(n-1)$-dimensional projective hyperplanes.
3. Algebraic sets of degree k in P^1: k-point sets.

Remark. A homogeneous equation of degree 1 is called a *homogeneous linear equation.*

Example. The equation $\varphi(x) = \alpha_0x_0 + \alpha_1x_1 + \ldots + \alpha_nx_n = 0$ is a homogeneous linear equation.

Theorem. (**A homogeneous linear equation of a line in** P^2)

Every line in P^2 has the equation $\alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2 = 0$, where $(\alpha_0, \alpha_1, \alpha_2) \neq (0,0,0)$.

Proof. If $\alpha_1 = \alpha_2 = 0$ and $\alpha_0 \neq 0$, then the equation $\alpha_0 x_0 = 0$ describes in P^2 the set of points of the form $\{0, x_1, x_2\}$, that is, an improper line.

If $(\alpha_1, \alpha_2) \neq (0,0)$, then for proper points $\{1, x_1, x_2\}$ the equation $\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 = 0$ describes a line in $\mathbb{R}^2$. The direction $\{0, x_1, x_2\}$ of that line is perpendicular to the vector $[\alpha_1, \alpha_2]$, that is, $\alpha_1 x_1 + \alpha_2 x_2 = 0$. Therefore a line in P^2 always can be described by the above equation. □

Theorem. (**A homogeneous linear equation of a plane in** P^3)

Every plane in P^3 has the equation $\alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0$, where $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \neq (0,0,0,0)$.

Proof. Similar to the proof of above theorem. □

Theorem. $L \subseteq P^2$ – a line, $a = \{a_0, a_1, a_2\}, b = \{b_0, b_1, b_2\} \in L$, $a \neq b$

Then

$$L : \begin{vmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ x_0 & x_1 & x_2 \end{vmatrix} = 0.$$

Proof. Easy. □

Theorem. $P \subseteq P^3$ – a plane, $a = \{a_0, a_1, a_2, a_3\}, b = \{b_0, b_1, b_2, b_3\}, c = \{c_0, c_1, c_2, c_3\} \in P$,

a, b, c do not lie on the same line in P^3

Then

$$P : \begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ c_0 & c_1 & c_2 & c_3 \\ x_0 & x_1 & x_2 & x_3 \end{vmatrix} = 0.$$

Proof. Easy. □

Remark. $P, Q \subseteq P^3$ – planes

Then $P \cap Q = L$ is a line (proper or improper). If $P : \alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0$ and $Q : \beta_0 x_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 = 0$, then

$$L : \begin{cases} \alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0, \\ \beta_0 x_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 = 0. \end{cases}$$

It is an *edge equation* of a line L in P^3.

Theorem. An algebraic set in P^n and its degree are projective invariants.

Proof. Similar to the proof of the fact that an algebraic set in $\mathbb{R}^n$ and its degree are affine invariants. $\square$

Theorem. (**On position of a line under an algebraic set of degree k in P^n**)

$L, F \subseteq P^n$, L – a line, F – an algebraic set of degree k

Then

$$L \subseteq F \quad \vee \quad 0 \leq \overline{\overline{L \cap F}} \leq k.$$

Proof. Similar to the proof of analogous theorem in $\mathbb{R}^n$. $\square$

Theorem. (**On making a polynomial homogeneous**)

For every polynomial $\varphi : \mathbb{R}^n \to \mathbb{R}$ of degree k there exists a homogeneous polynomial $\psi : \mathbb{R}^{n+1} \to \mathbb{R}$ of degree k such that

$$\psi(x_0, x_1, \ldots, x_n) = x_0^k \cdot \varphi\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right).$$

Proof. We have $\varphi(x_1, \ldots, x_n) = \sum\limits_{i_1,\ldots,i_n} \alpha_{i_1 \ldots i_n} x_1^{i_1} \cdot \ldots \cdot x_n^{i_n}$, where $i_1 + \ldots + i_n \leq k$ for any $i_1, \ldots, i_n$. Then

$$\begin{aligned}\psi(x_0, x_1, \ldots, x_n) &= x_0^k \cdot \varphi\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right)\\ &= \sum_{i_1,\ldots,i_n} \alpha_{i_1 \ldots i_n} x_0^k \cdot \frac{x_1^{i_1}}{x_0^{i_1}} \cdot \ldots \cdot \frac{x_n^{i_n}}{x_0^{i_n}}\\ &= \sum_{i_1,\ldots,i_n} \alpha_{i_1 \ldots i_n} x_0^{k-(i_1+\ldots+i_n)} \cdot x_1^{i_1} \cdot \ldots \cdot x_n^{i_n}.\end{aligned}$$

Let $i_0 = k - (i_1 + \ldots + i_n)$. Then

$$\psi(x_0, x_1, \ldots, x_n) = \sum_{i_1,\ldots,i_n} \alpha_{i_1 \ldots i_n} x_0^{i_0} \cdot x_1^{i_1} \cdot \ldots \cdot x_n^{i_n}$$

and $i_0 + i_1 + \ldots + i_n = k$, that is, ψ is a homogeneous polynomial. $\square$

Theorem. $F : \varphi(x_1, \ldots, x_n) = 0$ – an algebraic set of degree k in $\mathbb{R}^n$

Then $F^* : x_0^k \cdot \varphi\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) = 0$ is an algebraic set in P^n such that $\deg(F^*) \leq \deg(F)$ and $F^* \cap \mathbb{R}^n = F$.

Proof. Obviously $\deg(F^*) \leq \deg(F)$. Let $x_0 \neq 0$. Then

$$x_0^k \cdot \varphi\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) = 0 \;\Leftrightarrow\; \varphi\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) = 0 \;\Leftrightarrow\; \left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) = \{x_0, x_1, \ldots, x_n\} \in F.$$

Hence $F^* \cap \mathbb{R}^n = F$. $\square$

Remark. A set F^* is called a *complete* algebraic set or a *completion* of a set F.

Complete conics in P^2:

1. Complete parabola

$$P^* : x_2^2 - 2dx_0x_1 = 0 \ - \text{ the canonical equation of a complete parabola in } P^2.$$

If $x_0 = 0$, then $x_2^2 = 0$, that is, $x_2 = 0$. Hence $\{0, 1, 0\}$ is the only improper point of a parabola in $\mathbb{R}^2$ in canonical position.

Conclusion. A parabola has exactly one improper point.

2. Ellipse

$$E : \frac{x_1^2}{\alpha_1^2} + \frac{x_2^2}{\alpha_2^2} = x_0^2 \ - \text{ the canonical equation of an ellipse in } P^2.$$

If $x_0 = 0$, then $x_1 = x_2 = 0$. In the projective space P^2 there is no such point $\{0, 0, 0\}$. Hence there are no improper points satisfying above equation.

Conclusion. An ellipse has no improper points.

3. Complete hyperbola

$$H^* : \frac{x_1^2}{\alpha_1^2} - \frac{x_2^2}{\alpha_2^2} = x_0^2 \ - \text{ the canonical equation of a complete hyperbola in } P^2.$$

If $x_0 = 0$, then $\frac{x_1^2}{\alpha_1^2} - \frac{x_2^2}{\alpha_2^2} = 0$, that is, $x_1 = \alpha_1, x_2 = \alpha_2$ or $x_1 = -\alpha_1, x_2 = \alpha_2$. Hence $\{0, \alpha_1, \alpha_2\}$ and $\{0, -\alpha_1, \alpha_2\}$ are improper points of a hyperbola in $\mathbb{R}^2$ in canonical position.

Conclusion. A hyperbola has exactly two improper points.

Theorem. The number of improper points of an algebraic set is an affine invariant.

Proof. Follows from definition of a projective affine transformation. $\square$

Conclusion. (**Affine classification of conics**)

There are exactly three affine classes of conics: parabola, ellipse and hyperbola.

Theorem. (**Projective classification of conics**)

All conics belong to the same projective class.

Proof. It suffices to note that the projective transformation

$$\begin{cases} \overline{x}_0 = -\frac{1}{2}x_0 + \frac{1}{2}x_1, \\ \overline{x}_1 = \frac{1}{2}x_0 + \frac{1}{2}x_1, \\ \overline{x}_2 = x_2 \end{cases}$$

transforms the complete parabola $x_2^2 - x_0x_1 = 0$ onto the complete hyperbola $\overline{x}_1^2 - \overline{x}_2^2 = \overline{x}_0^2$, and the projective transformation

$$\begin{cases} \overline{\overline{x}}_0 = \overline{x}_1, \\ \overline{\overline{x}}_1 = \overline{x}_2, \\ \overline{\overline{x}}_2 = \overline{x}_0 \end{cases}$$

transforms the complete hyperbola $\overline{x}_1^2 - \overline{x}_2^2 = \overline{x}_0^2$ onto the ellipse $\overline{\overline{x}}_1^2 + \overline{\overline{x}}_2^2 = \overline{\overline{x}}_0^2$. $\square$

Other algebraic sets of degree 2 in P^2:

1. A 1-point set either has one improper point or does not have any.

2. A union of two proper parallel lines in $\mathbb{R}^2$ has exactly one improper point: a direction of these lines.

3. A union of two proper intersecting lines in $\mathbb{R}^2$ has exactly two improper points: directions of these lines.

4. A union of a proper line and the improper line has infinitely many improper points, which form the improper line.

Complete quadrics in P^3:

1. Ellipsoid

$$E : \frac{x_1^2}{\alpha_1^2} + \frac{x_2^2}{\alpha_2^2} + \frac{x_3^2}{\alpha_3^2} = x_0^2 \; - \text{ the canonical equation of an ellipsoid in } P^3.$$

If $x_0 = 0$, then $x_1 = x_2 = x_3 = 0$. In the projective space P^3 there is no such point $\{0, 0, 0, 0\}$. Hence there are no improper points satisfying above equation.

Conclusion. An ellipsoid has no improper points.

2. Complete hyperboloid of one sheet

$$H_1^* : \frac{x_1^2}{\alpha_1^2} + \frac{x_2^2}{\alpha_2^2} - \frac{x_3^2}{\alpha_3^2} = x_0^2 \; - \text{ the canonical equation of a complete hyperboloid of one sheet in } P^3.$$

If $x_0 = 0$, then $\frac{x_1^2}{\alpha_1^2} + \frac{x_2^2}{\alpha_2^2} - \frac{x_3^2}{\alpha_3^2} = 0$. This is an equation of some complete conic in the improper plane.

Conclusion. A hyperboloid of one sheet has infinitely many improper points, which together form a complete conic in the improper plane.

3. Complete hyperboloid of two sheets

$$H_2^*: \frac{x_1^2}{\alpha_1^2} + \frac{x_2^2}{\alpha_2^2} - \frac{x_3^2}{\alpha_3^2} = -x_0^2 \text{ -- the canonical equation of a complete hyperboloid of two sheets in } P^3.$$

If $x_0 = 0$, then $\frac{x_1^2}{\alpha_1^2} + \frac{x_2^2}{\alpha_2^2} - \frac{x_3^2}{\alpha_3^2} = 0$. This is an equation of some complete conic in the improper plane.

Conclusion. A hyperboloid of two sheets has infinitely many improper points, which together form a complete conic in the improper plane.

4. Complete elliptic paraboloid

$$PE^*: \frac{x_1^2}{\alpha_1^2} + \frac{x_2^2}{\alpha_2^2} = 2x_0x_3 \text{ -- the canonical equation of a complete elliptic paraboloid in } P^3.$$

If $x_0 = 0$, then $\frac{x_1^2}{\alpha_1^2} + \frac{x_2^2}{\alpha_2^2} = 0$, that is, $x_1 = 0$ and $x_2 = 0$. Hence $\{0, 0, 0, 1\}$ is the only improper point of an elliptic paraboloid in $\mathbb{R}^3$ in canonical position.

Conclusion. An elliptic paraboloid has exactly one improper point.

5. Complete hyperbolic paraboloid

$$PH^*: \frac{x_1^2}{\alpha_1^2} - \frac{x_2^2}{\alpha_2^2} = 2x_0x_3 \text{ -- the canonical equation of a complete hyperbolic paraboloid in } P^3.$$

If $x_0 = 0$, then $\frac{x_1^2}{\alpha_1^2} - \frac{x_2^2}{\alpha_2^2} = 0$, that is, $\frac{x_1}{\alpha_1} = \frac{x_2}{\alpha_2}$ or $\frac{x_1}{\alpha_1} = -\frac{x_2}{\alpha_2}$. These are equations of two lines in the improper plane.

Conclusion. A hyperbolic paraboloid has infinitely many improper points, which together form two projective lines in the improper plane.

Theorem. (**Affine classification of quadrics**)

There are exactly five affine classes of quadrics: ellipsoid, hyperboloid of one sheet, hyperboloid of two sheets, elliptic paraboloid and hyperbolic paraboloid.

Proof. A projective affine transformation $P^3 \to P^3$ transforms the improper plane onto itself. Hence a set of improper points of a quadric goes onto a set of improper points by such transformation. Thus ellipsoids, hyperboloids, elliptic paraboloids and hyperbolic paraboloids are not identical from the affine point of view (since they have different sets of improper points). Moreover a hyperboloid of one sheet and a hyperboloid of two sheets are different from the affine point of view, because the first one is a ruled set but the second one is not. □

Theorem. (**Projective classification of quadrics**)

There are exactly two projective classes of quadrics: to the first class belong ellipsoids, complete hyperboloids of two sheets and complete elliptic paraboloids; to the second class – complete hyperboloids of one sheet and complete hyperbolic paraboloids.

Proof. It suffices to remark that the projective transformation

$$\begin{cases} \overline{x}_0 = x_3, \\ \overline{x}_1 = x_1, \\ \overline{x}_2 = x_2, \\ \overline{x}_3 = x_0 \end{cases}$$

transforms the ellipsoid $x_1^2+x_2^2+x_3^2 = x_0^2$ onto the complete hyperboloid of two sheets $\overline{x}_1^2+\overline{x}_2^2-\overline{x}_3^2 = -\overline{x}_0^2$, and the projective transformation

$$\begin{cases} \overline{x}_0 = \frac{1}{2}x_0 + \frac{1}{2}x_3, \\ \overline{x}_1 = x_1, \\ \overline{x}_2 = x_2, \\ \overline{x}_3 = \frac{1}{2}x_0 - \frac{1}{2}x_3 \end{cases}$$

transforms the complete elliptic paraboloid $x_1^2+x_2^2 = x_0x_3$ onto the ellipsoid $\overline{x}_1^2+\overline{x}_2^2+\overline{x}_3^2 = \overline{x}_0^2$. Moreover the projective transformation

$$\begin{cases} \overline{x}_0 = \frac{1}{2}x_0 + \frac{1}{2}x_3, \\ \overline{x}_1 = x_1, \\ \overline{x}_2 = \frac{1}{2}x_0 - \frac{1}{2}x_3 \\ \overline{x}_3 = x_2, \end{cases}$$

transforms the complete hyperbolic paraboloid $x_1^2-x_2^2 = x_0x_3$ onto the complete hyperboloid of one sheet $\overline{x}_1^2+\overline{x}_2^2-\overline{x}_3^2 = \overline{x}_0^2$. Hence ellipsoids, complete hyperboloids of two sheets and complete elliptic paraboloids belong to the same projective class, and complete hyperboloids of one sheet and complete hyperbolic paraboloids also belong to the same projective class. These classes are different, since hyperboloids of one sheet and hyperboloids of two sheets cannot belong to the same projective class (the first have rectilinear generators and the second not). □

Remark. The first projective class consists of quadrics which are not ruled sets, and the second consists of ruled quadrics.

Other algebraic sets of degree 2 in P^3:

1. An elliptic cone

$$SE : \frac{x_1^2}{\alpha_1^2} + \frac{x_2^2}{\alpha_2^2} = x_3^2 \ - \text{ the canonical equation of an elliptic cone in } P^3.$$

If $x_0 = 0$, then $\frac{x_1^2}{\alpha_1^2} + \frac{x_2^2}{\alpha_2^2} = x_3^2$. This is an equation of some complete conic in the improper plane.

Conclusion. An elliptic cone has infinitely many improper points, which together form a complete conic in the improper plane.

2. A complete elliptic cylinder

$$WE^* : \frac{x_1^2}{\alpha_1^2} + \frac{x_2^2}{\alpha_2^2} = x_0^2 \text{ -- the canonical equation of a complete elliptic cylinder in } P^3.$$

If $x_0 = 0$, then $\frac{x_1^2}{\alpha_1^2} + \frac{x_2^2}{\alpha_2^2} = 0$, that is, $x_1 = 0$ and $x_2 = 0$. Hence $\{0, 0, 0, 1\}$ is the only improper point of an elliptic cylinder in $\mathbb{R}^3$ in canonical position.

Conclusion. An elliptic cylinder has exactly one improper point.

3. A complete parabolic cylinder

$$WP^* : x_2^2 = 2dx_0x_1 \text{ -- the canonical equation of a complete parabolic cylinder in } P^3.$$

If $x_0 = 0$, then $x_2 = 0$. This is an equation of a line in the improper plane.

Conclusion. A parabolic cylinder has infinitely many improper points, which together form a projective line in the improper plane.

4. A complete hyperbolic cylinder

$$WH^* : \frac{x_1^2}{\alpha_1^2} - \frac{x_2^2}{\alpha_2^2} = x_0^2 \text{ -- the canonical equation of a complete hyperbolic cylinder in } P^3.$$

If $x_0 = 0$, then $\frac{x_1^2}{\alpha_1^2} - \frac{x_2^2}{\alpha_2^2} = 0$, that is, $\frac{x_1}{\alpha_1} = \frac{x_2}{\alpha_2}$ or $\frac{x_1}{\alpha_1} = -\frac{x_2}{\alpha_2}$. These are equations of two lines in the improper plane.

Conclusion. A hyperbolic cylinder has infinitely many improper points, which together form two projective lines in the improper plane.

Moreover we have in P^3:

1. A 1-point set either has one improper point or does not have any.

2. A union of two proper parallel planes in $\mathbb{R}^3$ has infinitely many improper points, which together form a projective line in the improper plane.

3. A union of two proper nonparallel planes in $\mathbb{R}^3$ has infinitely many improper points, which together form two projective lines in the improper plane.

4. A union of a proper plane and the improper plane has infinitely many improper points, which form the improper plane.

Definition. (**Complex n-dimensional Cartesian space**)

$$\mathbb{C}^n \underset{df}{=} \{(z_1, \ldots, z_n) : z_1, \ldots, z_n \in \mathbb{C}\}.$$

Definition. $f : \mathbb{C}^n \to \mathbb{C}^n$ – a transformation

$f(z_1, \ldots, z_n) = (z'_1, \ldots, z'_n)$, where $z'_j = \alpha_{0j} + \alpha_{1j} z_1 + \ldots + \alpha_{nj} z_n$ for $j = 1, \ldots, n$

$A = [\alpha_{ij}],\ i, j = 1, \ldots, n$

A transformation f is called:

1) a *real isometry* $\underset{df}{\Leftrightarrow} \bigwedge_{i,j} \alpha_{ij} \in \mathbb{R}$ and A is orthogonal,

2) a *complex isometry* $\underset{df}{\Leftrightarrow} \bigwedge_{i,j} \alpha_{ij} \in \mathbb{C}$ and A is orthogonal,

3) a *real similarity with the ratio* $\lambda > 0$ $\underset{df}{\Leftrightarrow} \bigwedge_{i,j} \alpha_{ij} \in \mathbb{R}$ and $\frac{1}{\lambda}A$ is orthogonal,

4) a *complex similarity with the ratio* $\lambda > 0$ $\underset{df}{\Leftrightarrow} \bigwedge_{i,j} \alpha_{ij} \in \mathbb{C}$ and $\frac{1}{\lambda}A$ is orthogonal,

5) a *real affine transformation* $\underset{df}{\Leftrightarrow} \bigwedge_{i,j} \alpha_{ij} \in \mathbb{R}$ and A is nonsingular,

6) a *complex affine transformation* $\underset{df}{\Leftrightarrow} \bigwedge_{i,j} \alpha_{ij} \in \mathbb{C}$ and A is nonsingular.

Remark. The following can be extended without change to the complex space $\mathbb{C}^n$:

1) definitions of operations on points of the space and formal rules which apply to these operations,

2) the notion of a vector as an ordered pair of points,

3) arithmetical definitions of the equality of vectors, a free vector, operations on free vectors, a linear independence of free vectors,

4) the notions of a parallelism of vectors and their direction,

5) the notion of a perpendicularity of vectors.

Definition. $a, b \in \mathbb{C}^n$, $a \neq b$

A *complex line* in $\mathbb{C}^n$ is defined as a set of points of the form $x(t) = (1-t)a + tb$, where $t \in \mathbb{C}$.

Remark. Vectors which lie on the one line are parallel, their direction is called the direction of this line.

Theorem. $L \subseteq \mathbb{C}^n$ – a line, $a \in L$, $\mathfrak{a} \parallel L$, $\mathfrak{a} \neq 0$

Then

$$L : x(t) = a + t \cdot (\mathfrak{a}),\ \text{where } t \in \mathbb{C}.$$

Proof. The same like in $\mathbb{R}^n$. $\square$

Definition. $\mathfrak{a}, \mathfrak{b} \in \mathbb{C}^n$ – linearly independent vectors

A *complex plane* in $\mathbb{C}^n$ is defined as a set of all linear combinations of vectors $\mathfrak{a}$ and $\mathfrak{b}$.

Definition. $\mathfrak{a}_1, \dots, \mathfrak{a}_k \in \mathbb{C}^n$ – linearly independent vectors

A *complex k-dimensional hyperplane* in $\mathbb{C}^n$ is defined as a set of all linear combinations of vectors $\mathfrak{a}_1, \dots, \mathfrak{a}_k$.

Remark. Linear equations of a line in $\mathbb{C}^2$, a plane in $\mathbb{C}^3$ and $(n-1)$-dimensional hyperplane in $\mathbb{C}^n$ can be extended without change.

Remark. The following can be extended without change to complex space $\mathbb{C}^n$:

1) definition of an algebraic set of degree k,

2) the affine invariance of an algebraic set and its degree.

Definition. (**Complex n-dimensional projective space**)

The *complex n-dimensional projective space CP^n* is defined as a set of all ordered $(n+1)$-tuples $\{z_0, z_1, \dots, z_n\}$ of complex numbers, not all zero, where proportional systems are always considered as one and the same point.

Definition.

Proper points in $CP^n \underset{df}{=}$ points of the form $\{z_0, z_1, \dots, z_n\} = \left(\frac{z_1}{z_0}, \dots, \frac{z_n}{z_0}\right) \in \mathbb{C}^n$, where $z_0 \neq 0$.

Improper points in $CP^n \underset{df}{=}$ points of the form $\{0, z_1, \dots, z_n\} \in CP^n$.

Definition. $f : CP^n \overset{onto}{\longrightarrow} CP^n, \ \ \{z_0, z_1, \dots, z_n\} \in CP^n$

f is a *complex projective transformation* $\underset{df}{\Leftrightarrow}$ $f(z_0, z_1, \dots, z_n) = \{z'_0, z'_1, \dots, z'_n\}$, where $z'_j = \alpha_{0j} z_0 + \alpha_{1j} z_1 + \ldots + \alpha_{nj} z_n$ for $j = 0, 1, \dots, n$ and a matrix of f:

$$A_f = \begin{bmatrix} \alpha_{00} & \alpha_{10} & \dots & \alpha_{n0} \\ \alpha_{01} & \alpha_{11} & \dots & \alpha_{n1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{0n} & \alpha_{1n} & \dots & \alpha_{nn} \end{bmatrix}$$

is nonsingular.

If all $\alpha_{ij} \in \mathbb{R}$, then f is called a *real projective transformation.*

Remark. Directly from definition it is seen that projective transformations are one-to-one.

Remark. Similarly like in P^n every affine transformation (in particular, every isometry) $\mathbb{C}^n \to \mathbb{C}^n$ can be regarded as a projective transformation $CP^n \to CP^n$ such that the proper points are transformed onto the proper ones, and the improper points are transformed onto the improper ones.

Definition. $a = \{a_0, a_1, \ldots, a_n\}, b = \{b_0, b_1, \ldots, b_n\} \in CP^n,\ \ a \neq b$

A *complex projective line* in CP^n is defined as a set of points of the form $x(\lambda, \mu) = \{\lambda a_0 + \mu b_0, \lambda a_1 + \mu b_1, \ldots, \lambda a_n + \mu b_n\}$, where $(\lambda, \mu) \in \mathbb{C}^2 \backslash \{(0,0)\}$.

Theorem. Through every two different points of CP^n there passes exactly one complex projective line.

Proof. Similar to that in P^n. □

Conclusion. In the space CP^n a line that contains two improper points consists of improper points only (it is an improper line). Every line which contains at least one proper point (that is, it is a proper line) contains exactly one improper point.

Conclusion. The space CP^n can be obtained from the space $\mathbb{C}^n$ in a similar way to that in which the space P^n is obtained from the space $\mathbb{R}^n$.

Conclusion. Any two distinct lines of the plane CP^2 intersect at precisely one point (proper or improper).

Remark. The concepts of a complex projective plane and a complex projective k-dimensional hyperplane are defined in CP^n similarly like the concepts of a projective plane and a k-dimensional projective hyperplane in P^n.

Remark. To the space CP^n the following can be extended without change:

1) definition of an algebraic set of degree k,

2) the projective invariance of an algebraic set and its degree.

Conclusion. If $F : \varphi(x_1, \ldots, x_n) = 0$ is an algebraic set of degree k in $\mathbb{C}^n$, then $F^* : \psi(x_0, x_1, \ldots, x_n) = 0$ is a completion of a set F in CP^n, where $\psi(x_0, x_1, \ldots, x_n) = x_0^k \cdot \varphi\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right)$ is a homogeneous polynomial.

Theorem. (**On position of a line under an algebraic set of degree $k \geq 1$ in CP^n**)

$L, F \subseteq CP^n$, L – a line, F – an algebraic set of degree $k \geq 1$

Then

$$L \subseteq F \quad \vee \quad 1 \leq \overline{\overline{L \cap F}} \leq k.$$

Proof. Similar to the proof of analogous theorem in $\mathbb{R}^n$. □

Theorem. (**On completion**)

$F : \varphi(x_1, \ldots, x_n) = 0$ – an algebraic set of degree k in $\mathbb{C}^n$, $\psi(x_0, x_1, \ldots, x_n) = x_0^k \cdot \varphi\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) = 0$

Then $F^* : \psi(x_0, x_1, \ldots, x_n) = 0$ is an algebraic set of degree k in CP^n, which is obtained from F by adding its all improper points.

(without proof)

Algebraic sets of degree ≤ 2 in CP^n (P^n):

We know that $(n-1)$-dimensional hyperplanes (proper or improper) are algebraic sets of degree 1 in CP^n (P^n).

Now, let us consider algebraic sets of degree 2 in CP^n (P^n). Such sets are described by algebraic equations of degree 2 in which occurs a homogeneous polynomial of degree 2 called a quadratic form.

Recall that a *quadratic form* is a function φ such that

$$\varphi(x) = \sum_{i,j=0}^{n} \alpha_{ij} x_i x_j,$$

where $\alpha_{ij} = \alpha_{ji}$.

Then

$$\mathfrak{M}(\varphi) = [\alpha_{ij}],\ i,j = 0,1,\ldots,n\ -\ \text{the } \textit{great matrix} \text{ of a form } \varphi,$$

$$\mathfrak{m}(\varphi) = [\alpha_{ij}],\ i,j = 1,\ldots,n\ -\ \text{the } \textit{small matrix} \text{ of a form } \varphi,$$

$$\Delta(\varphi) = \det(\mathfrak{M}(\varphi))\ -\ \text{the } \textit{great discriminant} \text{ of a form } \varphi,$$

$$\delta(\varphi) = \det(\mathfrak{m}(\varphi))\ -\ \text{the } \textit{small discriminant} \text{ of a form } \varphi$$

and

$$\varphi_i(x) = \sum_{j=0}^{n} \alpha_{ij} x_j = \alpha_{i0}x_0 + \alpha_{i1}x_1 + \ldots + \alpha_{in}x_n - \text{the } \textit{ith derivative polynomial} \text{ of a form } \varphi \text{ for } i = 0,1,\ldots,n.$$

Thus

$$\varphi_i(x) = \frac{1}{2}\varphi'_{x_i}(x) = \frac{1}{2}\frac{\partial\varphi}{\partial x_i}(x) \text{ for } i = 0,1,\ldots,n,$$

where $\frac{\partial\varphi}{\partial x_i}$ means a partial derivative of φ, that is, a derivative of φ under a variable x_i.

Conclusion. F – an algebraic set of degree ≤ 2 in CP^n (P^n)

φ – a quadratic form, $\varphi(x) = \sum\limits_{i,j=0}^{n} \alpha_{ij}x_ix_j$

Then

$$F : \varphi(x) = 0 \quad \Rightarrow \quad F : \sum_{i=0}^{n} \varphi_i(x) \cdot x_i = 0.$$

Definition. F – an algebraic set of degree ≤ 2 in CP^n (P^n), L – a line

We know that $L \cap F = \emptyset \ \vee \ L \subset F \ \vee \ \overline{\overline{L \cap F}} = 1 \ \vee \ \overline{\overline{L \cap F}} = 2$. If $L \cap F = \{a\}$, then L is called the *line tangent to F at the point a.*

Theorem. A line tangent to an algebraic set of degree ≤ 2 is a projective invariant.

Proof. Follows directly from definition. □

Definition. F – an algebraic set of degree ≤ 2 in CP^n (P^n)

An *asymptote* of $F \underset{df}{=}$ a line tangent to F at an improper point.

Examples.
1. A parabola has one improper asymptote.
2. An ellipse does not have any asymptotes.
3. A hyperbola has two proper asymptotes.

Theorem. An asymptote of an algebraic set of degree ≤ 2 is an affine invariant.

Proof. Follows directly from definition. □

Definition. F – an algebraic set of degree ≤ 2 in CP^n (P^n), $a \in F$

$$S(a) \underset{df}{=} \text{ a union of all lines tangent to } F \text{ at a point } a.$$

Remark. We have:

$$S(a) = CP^n \ \vee \ S(a) \text{ is an } (n-1)\text{−dimensional hyperplane.}$$

Definition. F – an algebraic set of degree ≤ 2 in CP^n (P^n), $a \in F$

a is a *singular point* of $F \underset{df}{\Leftrightarrow} S(a) = CP^n$ (P^n),

a is a *regular point* of $F \underset{df}{\Leftrightarrow} S(a) = H^{n-1}$.

A *singular direction* of $F \underset{df}{=}$ a singular improper point of F.

Examples.
1. A line – every point is singular.
2. A pair of intersecting lines – the intersection point is singular.
3. A pair of parallel lines – the direction of these lines is a singular direction.
4. Conics – lack of singular points.
5. A cone – the vertex is singular.

6. A cylinder – the direction of a generator is singular.
7. Quadrics – lack of singular points.

Theorem. $F : \varphi(x) = 0, \;\; a \in F$

Then

a is singular $\Leftrightarrow \varphi_i(a) = 0$ for $i = 0, \ldots, n$,

a is regular $\Leftrightarrow \bigvee\limits_{0 \le i \le n} \varphi_i(a) \neq 0$,

where φ_i is the ith derivative polynomial of the form φ for $i = 0, 1, \ldots, n$.

Proof. $F : \varphi(x) = \sum\limits_{i,j=0}^{n} \alpha_{ij} x_i x_j = 0, \;\; \alpha_{ij} = \alpha_{ji}$

$a = \{a_0, a_1, \ldots, a_n\} \in F, \;\; b = \{x_0, x_1, \ldots, x_n\} \in CP^n, \;\; a \neq b$

Assume that a is singular, that is, every line which passes through a is tangent to F. Take the line

$$L(a, b) : x(\lambda, \mu) = \{\lambda a_0 + \mu x_0, \lambda a_1 + \mu x_1, \ldots, \lambda a_n + \mu x_n\}, \text{ where } (\lambda, \mu) \neq (0, 0).$$

Then the point a satisfies the equation

$$\sum_{i,j=0}^{n} \alpha_{ij} (\lambda a_i + \mu x_i)(\lambda a_j + \mu x_j) = 0,$$

which is equivalent to

$$\lambda^2 \sum_{i,j=0}^{n} \alpha_{ij} a_i a_j + 2\lambda\mu \sum_{i,j=0}^{n} \alpha_{ij} a_i x_j + \mu^2 \sum_{i,j=0}^{n} \alpha_{ij} x_i x_j = 0.$$

This means that there must be

$$\sum_{i,j=0}^{n} \alpha_{ij} a_i x_j = 0.$$

The above equation is equivalent to

$$\frac{\partial}{\partial x_j} \sum_{i,j=0}^{n} \alpha_{ij} a_i x_j = \sum_{i=0}^{n} \alpha_{ij} a_i = \varphi_j(a) = 0 \text{ for } j = 0, 1, \ldots, n.$$

Thus a is singular iff $\varphi_i(a) = 0$ for $i = 0, 1, \ldots, n$.
The second part follows directly from the first. □

Theorem.
F has at least one singular point $\Leftrightarrow \Delta(\varphi) = 0$.

Proof. $F : \varphi(x) = 0, \;\; \varphi$ – a quadratic form, $\;a \in F$

We have

a is a singular point of $F \Leftrightarrow \varphi_i(a) = 0$ for $i = 0, 1, \ldots, n \Leftrightarrow \sum_{j=0}^{n} \alpha_{ij} a_j = 0$ for $i = 0, 1, \ldots, n \Leftrightarrow$

$$\begin{cases} \alpha_{00}a_0 + \alpha_{01}a_1 + \ldots + \alpha_{0n}a_n = 0 \\ \alpha_{10}a_0 + \alpha_{11}a_1 + \ldots + \alpha_{1n}a_n = 0 \\ \quad\vdots \\ \alpha_{n0}a_0 + \alpha_{n1}a_1 + \ldots + \alpha_{nn}a_n = 0 \end{cases}$$

$\Leftrightarrow \det(\mathfrak{M}(\varphi)) = \Delta(\varphi) = 0.$ □

Conclusion.

F does not have any singular points $\Leftrightarrow \Delta(\varphi) \neq 0$.

Theorem. A singular point of an algebraic set of degree ≤ 2 in CP^n (P^n) is a projective invariant (so also an affine invariant).

Proof. Follows directly from definition. □

Theorem. A singular direction of an algebraic set of degree ≤ 2 in CP^n (P^n) is an affine invariant.

Proof. Follows directly from definition. □

Definition. F – an algebraic set of degree ≤ 2 in CP^n (P^n), $a \in F$ – a regular point of F

A *hyperplane tangent to F at a point a* $\underset{df}{=}$ $S(a) = H^{n-1}$.

Definition. F – an algebraic set of degree ≤ 2 in CP^n (P^n)

An *asymptotic hyperplane* of F $\underset{df}{=}$ a hyperplane tangent to F at an improper point.

Remark. If an improper point of F is also singular (so it is a singular direction), then there does not exist a hyperplane tangent to F at that point (there does not exist an asymptotic hyperplane).

Theorem. An asymptotic hyperplane of an algebraic set of degree ≤ 2 in CP^n (P^n) is an affine invariant.

Proof. Follows directly from definition. □

Definition. (**Polar**) F – an algebraic set of degree ≤ 2 in CP^n, $F : \varphi(x) = \sum_{i,j=0}^{n} \alpha_{ij} x_i x_j = 0$

$a \in CP^n$, a is not a singular point of F

Then

1) $a \in F$ (so it is regular)

We define

$$B(a) \underset{df}{=} S(a),$$

so it is a hyperplane tangent to F at the point a.

Then

$$B(a): \sum_{i,j=0}^{n} \alpha_{ij} a_i x_j = 0.$$

2) $a \notin F$, $\mathcal{W}$ – a bundle of all lines in CP^n which pass through a

Let $L \in \mathcal{W}$. Then $1 \leq \overline{\overline{L \cap F}} \leq 2$, that is, $L \cap F = \{p, q\}$, where $p = q$ or $p \neq q$. From every line $L \in \mathcal{W}$ we choose precisely one point x, which will belong to $B(a)$ in the following way:

$$\begin{aligned} p = q \ &\Rightarrow x = p = q, \\ p \neq q \ &\Rightarrow x \text{ is the fourth harmonic of points} \\ &\quad\ p, q, a, \text{ that is, } (p, q; a, x) = -1. \end{aligned}$$

Then the analytic formula of $B(a)$ is as follows

$$B(a): \sum_{i=0}^{n} \varphi_i(a) \cdot x_i = 0$$

or

$$B(a): \sum_{i=0}^{n} \varphi'_{x_i}(a) \cdot x_i = 0.$$

We call $B(a)$ the *polar of the point a with respect to F*, and point a – the *pole of $B(a)$ with respect to F*.

Remark. Since $\varphi_i(x) = \sum\limits_{j=0}^{n} \alpha_{ij} x_j$, so

$$\sum_{i=0}^{n} \varphi_i(a) \cdot x_i = \sum_{i=0}^{n} \left(\sum_{j=0}^{n} \alpha_{ij} a_j \right) x_i = \sum_{j=0}^{n} \left(\sum_{i=0}^{n} \alpha_{ij} x_i \right) a_j = \sum_{j=0}^{n} \varphi_j(x) \cdot a_j.$$

Thus

$$B(a): \sum_{i=0}^{n} \varphi_i(x) \cdot a_i = 0.$$

Theorem. A polar of a point a with respect to an algebraic set of degree ≤ 2 in CP^n is a projective invariant.

Proof. Follows directly from definition. $\square$

Definition. (**Diametral hyperplane**) F – an algebraic set of degree ≤ 2 in CP^n

A *diametral hyperplane* of $F \underset{df}{=}$ a polar of an improper point.

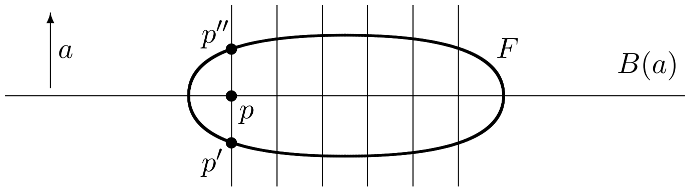
If a is an improper point, then $B(a)$ is a *diametral hyperplane of F conjugate to the direction a.*

Remark. The diametral hyperplane $B(a)$ is not defined when the direction a is singular.

Remark. If the improper point a belongs to F, then a diametral hyperplane of F conjugate to the direction a is an asymptotic hyperplane.

Theorem. F – an algebraic set of degree ≤ 2 in CP^n, $a \notin F$, a – improper

Then a diametral hyperplane $B(a)$ passes through centres of strings with direction a:



Proof. Since $a \notin F$, it follows that $B(a)$ is a proper hyperplane. Any line L with direction a intersects F at at most 2 points p' and p''. If $p' \neq p''$, then L intersects $B(a)$ at such point p that $(a, p; p', p'') = -1$. Since a is improper, it follows that p must be a centre of a segment $\langle p', p'' \rangle$. $\square$

Theorem. A diametral hyperplane of an algebraic set of degree ≤ 2 in CP^n is an affine invariant.

Proof. Follows directly from definition. $\square$

Definition. (**A centre of an algebraic set of degree** ≤ 2)

F – an algebraic set of degree ≤ 2 in CP^n (P^n), $a \in CP^n$ (P^n)

a is a *centre* of $F \underset{df}{\Leftrightarrow}$ a is a singular point of F or a is a pole of an improper hyperplane.

Conclusion. A centre of an algebraic set of degree ≤ 2 is an affine invariant.

Theorem. $F : \varphi(x) = \sum\limits_{i,j=0}^{n} \alpha_{ij} x_i x_j = 0$

Then

a is a centre of $F \Leftrightarrow \varphi_i(a) = 0$ for $i = 1, \ldots, n$.

Proof. ($\Rightarrow$) If a is singular, then $\varphi_i(a) = 0$ for $i = 0, 1, \ldots, n$. So we get a thesis. If a is a pole of an improper hyperplane, then from the form of the equation of the polar $B(a)$ we get the thesis.

($\Leftarrow$) Assume that $\varphi_i(a) = 0$ for $i = 1, \ldots, n$. If $\varphi_0(a) = 0$, then a is singular. If $\varphi_0(a) \neq 0$, then from the form of the equation of the polar $B(a)$ we get that a is a pole of an improper hyperplane. $\square$

Theorem. Proper centres of an algebraic set F which belong to F are singular points of F.

Proof. Let $F : \sum\limits_{i=0}^{n} \varphi_i(x) \cdot x_i = 0$, where φ_i is the ith derivative polynomial of the form φ for $i = 0, 1, \ldots, n$. If $a = \{a_0, a_1, \ldots, a_n\}$ is a proper centre of F and $a \in F$, then $\sum\limits_{i=0}^{n} \varphi_i(a) \cdot a_i = 0$ and $\varphi_i(a) = 0$ for $i = 1, \ldots, n$. Hence $\varphi_0(a) \cdot a_0 = 0$. Since a is proper, $a_0 \neq 0$. Hence $\varphi_0(a) = 0$. Thus $\varphi_i(a) = 0$ for $i = 0, 1, \ldots, n$, that is, a is a singular point of F. $\square$

Theorem. A proper centre of F is its centre of symmetry. If F does not contain any improper hyperplane, then the converse is also true.

Proof. Let $F : \varphi(x) = \sum\limits_{i,j=0}^{n} \alpha_{ij} x_i x_j = 0$. Assume that $c = \{c_0, c_1, \ldots, c_n\}$, where $c_0 = 1$, is a centre of F. Let $a = \{a_0, a_1, \ldots, a_n\}$ and $a' = \{a'_0, a'_1, \ldots, a'_n\}$ be points symmetric about c, where $a_0 = a'_0 = 1$. Then $a'_i = 2c_i - a_i$ for $i = 0, 1, \ldots, n$. We will show that if $a \in F$, then $a' \in F$. We have

$$\begin{aligned}
\sum_{i,j=0}^{n} \alpha_{ij} a'_i a'_j &= \sum_{i,j=0}^{n} \alpha_{ij} (2c_i - a_i)(2c_j - a_j) \\
&= 4 \sum_{j=0}^{n} \left(\sum_{i=0}^{n} \alpha_{ij} c_i \right) c_j - 4 \sum_{j=0}^{n} \left(\sum_{i=0}^{n} \alpha_{ij} c_i \right) a_j + \sum_{i,j=0}^{n} \alpha_{ij} a_i a_j \\
&= 0,
\end{aligned}$$

since $\varphi_j(c) = \sum\limits_{i=0}^{n} \alpha_{ij} c_i = 0$ for $j = 1, \ldots, n$, $c_0 = a_0 = 1$ and $\sum\limits_{i,j=0}^{n} \alpha_{ij} a_i a_j = 0$. Thus $a' \in F$, that is, c is a centre of symmetry of F.

Now assume that F does not contain an improper hyperplane H_∞ and that $c = \{c_0, c_1, \ldots, c_n\}$, where $c_0 = 1$, is not a centre of F. We will show that c is not a centre of symmetry of F. Since $H_\infty \nsubseteq F$, the equation $\sum\limits_{i,j=1}^{n} \alpha_{ij} y_i y_j = 0$ describes in H_∞ some algebraic set F' of degree 1 or 2. Moreover c is not a centre of F, whence the equation $\sum\limits_{j=1}^{n} \left(\sum\limits_{i=1}^{n} \alpha_{ij} c_i \right) y_j = 0$ describes in H_∞ some $(n-2)$-dimensional hyperplane H'. Let $a \in H_\infty \backslash F'$ and $L' \subseteq H_\infty$ be a line such that $a \in L'$ and $L' \nsubseteq H'$. We have $\overline{\overline{L' \cap F'}} \leq 2$ and $\overline{\overline{L' \cap H'}} \leq 1$. Hence there is $b \in (L' \backslash F') \backslash H'$. Let $b = \{b_0, b_1, \ldots, b_n\}$, where $b_0 = 0$. Thus we have

$$\sum_{i,j=1}^{n} \alpha_{ij} b_i b_j \neq 0 \neq \sum_{j=1}^{n} \left(\sum_{i=0}^{n} \alpha_{ij} c_i \right) b_j.$$

Take a line L such that $c \in L$ and $b \parallel L$. Hence

$$L : x(t) = (c_0 + tb_0, c_1 + tb_1, \ldots, c_n + tb_n).$$

The intersection point of L and F we find from the equation

$$\sum_{i,j=0}^{n} \alpha_{ij}(c_i + tb_i)(c_j + tb_j) = 0,$$

that is,

$$\sum_{i,j=0}^{n} \alpha_{ij}c_ic_j + 2t\sum_{j=1}^{n}\left(\sum_{i=0}^{n}\alpha_{ij}c_i\right)b_j + t^2\sum_{i,j=1}^{n}\alpha_{ij}b_ib_j = 0.$$

The above has two roots t' and t'' such that $t' + t'' \neq 0$. Thus $L \cap F = \{x(t'), x(t'')\}$ and $x(t')$ and $x(t'')$ are not symmetric about c, since

$$\frac{x(t') + x(t'')}{2} = \left(c_0, c_1 + \frac{t' + t''}{2}b_1, \ldots, c_n + \frac{t' + t''}{2}b_n\right) \neq c.$$

So c is not a centre of symmetry of F. $\square$

Definition. F – an algebraic set of degree ≤ 2

A *special direction* of $F \underset{df}{=}$ an improper centre of F.

Conclusion. A singular direction of an algebraic set of degree ≤ 2 is a special one.

Conclusion. A special direction of an algebraic set of degree ≤ 2 is an affine invariant.

Theorem. Improper centres of an algebraic set F belong to F.

Proof. Let $F : \varphi(x) = 0$, where φ is a quadratic form. Then $F : \sum\limits_{i=0}^{n}\varphi_i(x) \cdot x_i = 0$, where φ_i is the ith derivative polynomial of the form φ for $i = 0, 1, \ldots, n$. Let $a = \{a_0, a_1, \ldots, a_n\}$ be an improper centre of F. Then $\varphi_i(a) = 0$ for $i = 1, \ldots, n$. We want to show that $\sum\limits_{i=0}^{n}\varphi_i(a) \cdot a_i = 0$. We have

$$\sum_{i=0}^{n}\varphi_i(a) \cdot a_i = \varphi_0(a) \cdot a_0 + \sum_{i=1}^{n}\varphi_i(a) \cdot a_i.$$

Since a is improper, we have $a_0 = 0$. Since a is a centre, we also have $\varphi_i(a) = 0$ for $i = 1, \ldots, n$. Hence

$$\varphi_0(a) \cdot a_0 + \sum_{i=1}^{n}\varphi_i(a) \cdot a_i = \varphi_0(a) \cdot 0 + \sum_{i=1}^{n} 0 \cdot a_i = 0.$$

Thus $a \in F$. $\square$

Theorem. If $F : \varphi(x) = 0$, then F has at least one special direction $\Leftrightarrow \delta(\varphi) = 0$.

Proof. Let $F : \varphi(x) = 0$, where φ is a quadratic form. Let $a = \{0, a_1, \dots, a_n\}$. Then a is a special direction of $F \Leftrightarrow \varphi_i(a) = 0$ for $i = 1, \dots, n \Leftrightarrow \sum\limits_{j=0}^{n} \alpha_{ij} a_j = 0$ for $i = 1, \dots, n \Leftrightarrow$

$$\begin{cases} \alpha_{11}a_1 + \alpha_{12}a_2 + \ldots + \alpha_{1n}a_n = 0 \\ \alpha_{21}a_1 + \alpha_{22}a_2 + \ldots + \alpha_{2n}a_n = 0 \\ \quad \vdots \\ \alpha_{n1}a_1 + \alpha_{n2}a_2 + \ldots + \alpha_{nn}a_n = 0 \end{cases} \Leftrightarrow$$

$\Leftrightarrow \det(\mathfrak{m}(\varphi)) = \delta(\varphi) = 0.$ $\square$

Theorem. If $F : \varphi(x) = 0$, then F has precisely one proper centre $\Leftrightarrow \delta(\varphi) \neq 0$.

Proof. Let $F : \varphi(x) = 0$, where φ is a quadratic form. Let $a = \{a_0, a_1, \dots, a_n\}$, where $a_0 = 1$. Then a is a proper centre of $F \Leftrightarrow \varphi_i(a) = 0$ for $i = 1, \dots, n \Leftrightarrow \sum\limits_{j=0}^{n} \alpha_{ij} a_j = 0$ for $i = 1, \dots, n \Leftrightarrow$

$$\begin{cases} \alpha_{10} + \alpha_{11}a_1 + \ldots + \alpha_{1n}a_n = 0 \\ \alpha_{20} + \alpha_{21}a_1 + \ldots + \alpha_{2n}a_n = 0 \\ \quad \vdots \\ \alpha_{n0} + \alpha_{n1}a_1 + \ldots + \alpha_{nn}a_n = 0 \end{cases} \Leftrightarrow$$

$$\begin{cases} \alpha_{11}a_1 + \ldots + \alpha_{1n}a_n = -\alpha_{10} \\ \alpha_{21}a_1 + \ldots + \alpha_{2n}a_n = -\alpha_{20} \\ \quad \vdots \\ \alpha_{n1}a_1 + \ldots + \alpha_{nn}a_n = -\alpha_{n0} \end{cases} \Leftrightarrow$$

$\Leftrightarrow \det(\mathfrak{m}(\varphi)) = \delta(\varphi) \neq 0.$ $\square$

Conclusion. Every algebraic set of degree ≤ 2 has at least one centre (proper or improper).

Examples.

1. For an ellipse $E : \varphi(x) = \alpha_2^2 x_1^2 + \alpha_1^2 x_2^2 - \alpha_1^2 \alpha_2^2 x_0^2 = 0$, where $\alpha_1, \alpha_2 > 0$ we have

$$\delta(\varphi) = \begin{vmatrix} \alpha_2^2 & 0 \\ 0 & \alpha_1^2 \end{vmatrix} \neq 0.$$

Thus an ellipse has precisely one proper centre. Similarly for a hyperbola.

2. For a parabola $P^* : \varphi(x) = x_2^2 - 2dx_0x_1 = 0$ we have

$$\delta(\varphi) = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0.$$

So a parabola has at least one improper centre, which must belong to a parabola. Since a parabola has precisely one improper point, it has precisely one improper centre. Similar situation occurs for an elliptic paraboloid and a hyperbolic paraboloid.

Remark. That can be shown similarly for other algebraic sets of degree ≤ 2.

Remark. If an algebraic set of degree ≤ 2 does not have a proper centre (so a centre of symmetry), then it can have at least one vertex which belongs to the intersection of this set and its hyperplane of symmetry.

Theorem. If an algebraic set F has a special direction, then the number of centres of symmetry of F is different from 1.

Proof. Let $F : \varphi(x) = 0$, where φ is a quadratic form. Since F has a special direction, then $\delta(\varphi) = 0$. Hence it is not true that $\delta(\varphi) \neq 0$, that is, F does not have precisely one centre of symmetry. $\square$

Theorem. A diametral hyperplane of an algebraic set F contains all centres of F.

Proof. Let $F : \varphi(x) = 0$, where φ is a quadratic form. Let $B(a)$ be a diametral hyperplane of F, where $a = \{a_0, a_1, \ldots, a_n\}$. Let b be a centre of F. We have $B(a) : \sum_{i=0}^{n} \varphi_i(x) \cdot a_i = 0$ and $\varphi_i(b) = 0$ for $i = 1, \ldots, n$. We want to show that $b \in B(a)$, that is,

$$\sum_{i=0}^{n} \varphi_i(b) \cdot a_i = 0.$$

But

$$\sum_{i=0}^{n} \varphi_i(b) \cdot a_i = \varphi_0(b) \cdot 0 + \sum_{i=1}^{n} \varphi_i(b) \cdot a_i = 0,$$

because $\varphi_i(b) = 0$ for $i = 1, \ldots, n$. Thus $b \in B(a)$. $\square$

Definition. (**Principal direction**)

F – an algebraic set of degree ≤ 2 in CP^n (P^n), $a = \{0, a_1, \ldots, a_n\}$

a is a *principal direction* of F $\underset{df}{\Leftrightarrow}$ 1) $B(a)$ doesn't exist $\vee$ 2) $B(a)$ is improper $\vee$ 3) $B(a) \perp a$.

Then:

1) a is a singular direction,
2) a is a special direction,
3) a is called a *nonspecial principal direction.*

Theorem. (**On principal directions**) $F : \varphi(x) = 0$, $a = \{0, a_1, \ldots, a_n\}$

Then a is a principal direction of $F \Leftrightarrow$

$$\bigvee_{\lambda} \varphi_i(a) = \lambda a_i, \quad i = 1, \ldots, n.$$

Moreover,

$\lambda = 0 \;\Leftrightarrow\; a$ is a special direction,

$\lambda \neq 0 \;\Leftrightarrow\; a$ is a nonspecial direction.

Proof. We know that $B(a) : \sum_{i=0}^{n} \varphi_i(a) \cdot x_i = 0$. If a isn't a special direction, then not all $\varphi_i(a)$ for $i = 1, \ldots, n$ are equal to 0, that is, $B(a)$ is proper. Further

$$B(a) \perp a \;\Leftrightarrow\; \bigvee_{\lambda} \varphi_i(a) = \lambda a_i, \quad i = 1, \ldots, n$$

(that is, $\varphi_i(a)$ are proportional to a_i for $i = 1, \ldots, n$). It is easy to see that

$\lambda = 0 \;\Leftrightarrow\; a$ is a special direction, and
$\lambda \neq 0 \;\Leftrightarrow\; a$ is a nonspecial direction. □

Conclusion. For every algebraic set of degree 2 there exists at least one principal direction.

Theorem. An $(n-1)$-dimensional proper hyperplane perpendicular to a singular direction of an algebraic set F of degree ≤ 2 is its hyperplane of symmetry (in P^n, CP^n).

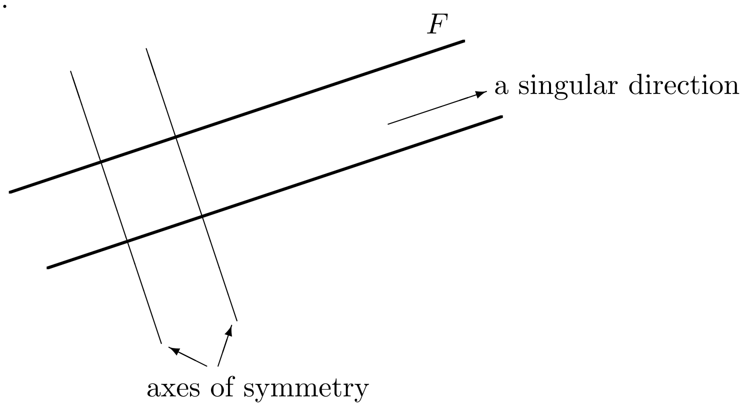
Proof. Let $F : \sum_{i,j=0}^{n} \alpha_{ij} x_i x_j = 0$. Let $a = \{0, a_1, \ldots, a_n\}$ be a singular direction of F, that is, a is special, so $a \in F$. Let H be an $(n-1)$-dimensional proper hyperplane such that $a \perp H$. Let $b = \{b_0, b_1, \ldots, b_n\}$, $b' = \{b'_0, b'_1, \ldots, b'_n\}$, where $b_0 = b'_0 = 1$, be points symmetric about H. Assume that $b \in F$, that is, $\sum_{i,j=0}^{n} \alpha_{ij} b_i b_j = 0$. We have $\{0, b_1 - b'_1, \ldots, b_n - b'_n\} \perp H$. Hence $b_i - b'_i = a_i$ for $i = 1, \ldots, n$, that is, $b'_i = b_i - a_i$. Thus

$$\sum_{i,j=0}^{n} \alpha_{ij} b'_i b'_j = \sum_{i,j=0}^{n} \alpha_{ij} (b_i - a_i)(b_j - a_j) = \sum_{i,j=0}^{n} \alpha_{ij} b_i b_j - 2 \sum_{i,j=0}^{n} \alpha_{ij} a_i b_j + \sum_{i,j=0}^{n} \alpha_{ij} a_i a_j$$
$$= 0 - 2 \sum_{i,j=0}^{n} \alpha_{ij} a_i b_j + 0 = -2 \sum_{j=0}^{n} \left(\sum_{i=0}^{n} \alpha_{ij} a_i \right) b_j = -2 \sum_{j=0}^{n} \varphi_j(a) \cdot b_j = 0.$$

Thus $b' \in F$. Hence H is a hyperplane of symmetry of F. □

Conclusion. A line perpendicular to a singular direction of an algebraic set of degree ≤ 2 in P^2 is its axis of symmetry.

Example.



Definition. (Principal diametral hyperplane)

A *principal diametral hyperplane* of a set $F \underset{df}{=}$ a diametral hyperplane of F conjugate to a nonspecial principal direction.

Theorem. A principal diametral hyperplane of a set F is a hyperplane of symmetry of F.

Proof. Let $F : \sum\limits_{i,j=0}^{n} \alpha_{ij} x_i x_j = 0$. Let $a = \{a_0, a_1, \ldots, a_n\}, a' = \{a'_0, a'_1, \ldots, a'_n\}$, where $a_0 = a'_0 = 1$, be points symmetric about the principal diametral hyperplane $B(b) : \sum\limits_{i=0}^{n} \varphi_i(b) \cdot x_i = 0$. Hence $b \perp B(b)$ and $\{0, a_1 - a'_1, \ldots, a_n - a'_n\} \perp B(b)$. Putting $b_0 = 0$ we can set $a_i - a'_i = b_i$, that is, $a'_i = a_i - b_i$ for $i = 0, 1, \ldots, n$. Assume that $a \in F$, that is, $\sum\limits_{i,j=0}^{n} \alpha_{ij} a_i a_j = 0$. Hence

$$\sum_{i,j=0}^{n} \alpha_{ij} a'_i a'_j = \sum_{i,j=0}^{n} \alpha_{ij} (a_i - b_i)(a_j - b_j) = \sum_{i,j=0}^{n} \alpha_{ij} a_i a_j - 2 \sum_{i,j=0}^{n} \alpha_{ij} a_i b_j + \sum_{i,j=0}^{n} \alpha_{ij} b_i b_j$$
$$= 0 + \sum_{i,j=0}^{n} \alpha_{ij} (b_i - 2a_i) b_j = \sum_{i=0}^{n} \left(\sum_{j=0}^{n} \alpha_{ij} b_j \right) (b_i - 2a_i) = \sum_{i=0}^{n} \varphi_i(b)(a_i - a'_i - 2a_i)$$
$$= -\sum_{i=0}^{n} \varphi_i(b)(a_i + a'_i) = 0,$$

since

$$\{a_0 + a'_0, a_1 + a'_1, \ldots, a_n + a'_n\} = \left\{1, \frac{a_1 + a'_1}{2}, \ldots, \frac{a_n + a'_n}{2}\right\} \in B(b)$$

(as the centre of the segment $\langle a, a' \rangle$).
Hence $a' \in F$. Thus $B(b)$ is the hyperplane of symmetry of F. $\square$

Definition.

The equation $\sum\limits_{i,j=0}^{n} \alpha_{ij} x_i x_j = 0$ has a *canonical form of the first kind* $\underset{df}{\Leftrightarrow}$ it has the form

$$\sum_{i=0}^{k} \alpha_i x_i^2 = 0, \text{ where } \alpha_i = \alpha_{ii} \neq 0 \text{ for } i = 1, \ldots, k \text{ and some } k = 0, 1, \ldots, n.$$

Remark. The canonical equations of an ellipse, a hyperbola, an ellipsoid, a hyperboloid of one sheet, a hyperboloid of two sheets, an elliptic cylinder, a hyperbolic cylinder and a cone have a canonical form of the first kind.

Definition.

The equation $\sum\limits_{i,j=0}^{n} \alpha_{ij} x_i x_j = 0$ has a *canonical form of the second kind* $\underset{df}{\Leftrightarrow}$ it has the form

$$\sum_{i=1}^{k} \alpha_i x_i^2 + 2x_0 x_n = 0, \text{ where } \alpha_i = \alpha_{ii} \neq 0 \text{ for } i = 1, \ldots, k \text{ and some } k = 0, 1, \ldots, n-1$$

(instead of x_n there can be any other unknown which is not squared).

Remark. The canonical equations of a parabola, a parabolic cylinder, an elliptic paraboloid and a hyperbolic paraboloid have a canonical form of the second kind.

Definition. $F : \sum\limits_{i,j=0}^{n} \alpha_{ij} x_i x_j = 0$ – an algebraic set of degree ≤ 2 in $\mathbb{C}^n$ (CP^n)

The set F is called *real* $\underset{df}{\Leftrightarrow}$ $\alpha_{ij} \in \mathbb{R}$ for every $i, j = 0, 1, \ldots, n$.

Theorem. (**On reduction**)

For every algebraic set $F : \sum\limits_{i,j=0}^{n} \alpha_{ij} x_i x_j = 0$ in CP^n there exists an affine transformation f which transforms the set F onto a set defined by an equation in a canonical form. The canonical form is of the first kind if the set F has at least one proper centre, and it is of the second kind if the set F does not have any proper centre. If the set F is real, then it is always possible to choose a real isometry for the transformation f.

(without proof)

$F : \varphi(x) = 0$ – an algebraic set of degree ≤ 2 in CP^n (P^n)

Take

$$K(F) \underset{df}{=} r(\mathfrak{M}(\varphi)) \quad - \quad \text{the number of nonzero eigenvalues of } \mathfrak{M}(\varphi),$$

$$k(F) \underset{df}{=} r(\mathfrak{m}(\varphi)) \quad - \quad \text{the number of nonzero eigenvalues of } \mathfrak{m}(\varphi),$$

$L(F) \underset{df}{=}$ the absolute value of the difference of numbers of positive and negative eigenvalues of $\mathfrak{M}(\varphi)$,

$l(F) \underset{df}{=}$ the absolute value of the difference of numbers of positive and negative eigenvalues of $\mathfrak{m}(\varphi)$.

Theorem.

1. Two algebraic sets F and F' of degree ≤ 2 in CP^n are identical from the projective point of view $\Leftrightarrow K(F) = K(F')$.

2. Two real algebraic sets F and F' of degree ≤ 2 in CP^n are identical from the projective point of view $\Leftrightarrow K(F) = K(F')$ and $L(F) = L(F')$.

(without proof)

Conclusion. In CP^n there exists precisely 1 projective class of algebraic sets of degree 2 without singular points.

Conclusion. In CP^n there exist precisely n projective classes of all algebraic sets of degree 2.

Conclusion. In CP^n there exist precisely $E\left(\frac{n+3}{2}\right)$ projective classes of real algebraic sets of degree 2 without singular points[1].

Conclusion. In CP^n there exist precisely $\sum_{k=1}^{n} E\left(\frac{k+3}{2}\right)$ projective classes of all real algebraic sets of degree 2.

Conclusions.

1. In CP^2 there exist 2 projective classes of real algebraic sets of degree 2 without singular points: conics and algebraic sets without real points; and 4 projective classes of all real algebraic sets of degree 2: conics, algebraic sets without real points, pairs of real lines and pairs of imaginary lines which intersect at a real point.

2. In P^2 there exists 1 projective class of real algebraic sets of degree 2 without singular points: conics; and 2 projective classes of all real algebraic sets of degree 2: conics and pairs of real lines.

3. In CP^3 there exist 3 projective classes of real algebraic sets of degree 2 without singular points: quadrics which are ruled sets, quadrics which are not ruled sets and algebraic sets without real points; and 7 projective classes of all real algebraic sets of degree 2.

4. In P^3 there exist 2 projective classes of real algebraic sets of degree 2 without singular points: quadrics which are ruled sets and quadrics which are not ruled sets; and 5 projective classes of all real algebraic sets of degree 2: quadrics which are ruled sets, quadrics which are not ruled sets, cones, cylinders and pairs of real planes.

Theorem.

1. Two algebraic sets F and F' of degree ≤ 2 in CP^n are identical from the affine point of view $\Leftrightarrow K(F) = K(F')$ and $k(F) = k(F')$.

2. Two real algebraic sets F and F' of degree ≤ 2 in CP^n are identical from the affine point of view $\Leftrightarrow K(F) = K(F')$, $L(F) = L(F')$, $k(F) = k(F')$ and $l(F) = l(F')$.

(without proof)

[1] For $x \in \mathbb{R}$, the symbol $E(x)$ means the integer k such that $k \leq x < k+1$.

Conclusion. In CP^n there exist precisely 2 affine classes of algebraic sets of degree 2 without singular points.

Conclusion. In CP^n there exist precisely $3n-1$ affine classes of all algebraic sets of degree 2.

Conclusion. In CP^n there exist precisely $n+E\left(\frac{n+1}{2}\right)+1$ affine classes of real algebraic sets of degree 2 without singular points.

Conclusion. In CP^n there exist precisely n^2+3n-1 affine classes of all real algebraic sets of degree 2.

Affine classification of real algebraic sets of degree ≤ 2 in P^2:

Affine class	Improper points	Singular points	Centres
Ellipse	0	0	1 proper
Hyperbola	2	0	1 proper
Parabola	1	0	1 improper
Pair of proper intersecting lines	2	1 proper	1 proper
Pair of proper parallel lines	1	1 improper	proper line
Proper line + improper line	improper line	1 improper	1 improper
Proper line	1	proper line	proper line
Improper line	improper line	improper line	improper line

Remark. As we see, in P^2 there are 3 affine classes of algebraic sets of degree 2 without singular points. We know that in CP^2 there are $n+E\left(\frac{n+1}{2}\right)+1=4$ such classes: we additionally have imaginary algebraic set without singular points.

Remark. In P^2 there are 6 affine classes of all algebraic sets of degree 2. We know that in CP^2 there are $n^2+3n-1=9$ such classes: we additionally have imaginary algebraic set without singular points, pair of imaginary lines which intersect at a real proper point and pair of parallel imaginary lines.

Affine classification of real algebraic sets of degree ≤ 2 in P^3:

Affine class	**Improper points**	**Singular points**	**Centres**	**Remarks**
Ellipsoid	0	0	1 proper	nonruled set
Hyperboloid of one sheet	conic	0	1 proper	ruled set
Hyperboloid of two sheets	conic	0	1 proper	nonruled set
Elliptic paraboloid	1	0	1 improper	nonruled set
Hyperbolic paraboloid	two lines	0	1 improper	ruled set
Cone	conic	1 proper	1 proper	ruled set
Elliptic cylinder	1	1 improper	proper line	ruled set
Parabolic cylinder	one line	1 improper	improper line	ruled set
Hyperbolic cylinder	two lines	1 improper	proper line	ruled set
Pair of proper nonparallel planes	two lines	proper line	proper line	ruled set
Pair of proper parallel planes	one line	improper line	proper plane	ruled set
Proper plane + improper plane	improper plane	improper line	improper line	ruled set
Proper line	1	proper line	proper line	ruled set
Improper line	improper line	improper line	improper line	ruled set
Proper plane	one line	proper plane	proper plane	ruled set
Improper plane	improper plane	improper plane	improper plane	ruled set

Remark. As we see, in P^3 there are 5 affine classes of algebraic sets of degree 2 without singular points. We know that in CP^3 there are $n + E\left(\frac{n+1}{2}\right) + 1 = 6$ such classes: we additionally have imaginary algebraic set without singular points.

Remark. In P^3 there are 14 affine classes of all algebraic sets of degree 2. We know that in CP^3 there are $n^2 + 3n - 1 = 17$ such classes: we additionally have imaginary algebraic set without singular points, pair of imaginary planes which intersect at a real proper line and pair of parallel imaginary planes.

Conclusion. (**method of finding principal directions, hyperplanes of symmetry and a canonical equation of an algebraic set of degree** 2)

$F : \varphi(x) = 0$ in P^2 or P^3

1. We determine eigenvalues λ_i and eigenvectors $\mathfrak{a}_i$ of the matrix $\mathfrak{m}(\varphi)$.

2. If $\lambda_i = 0$, then $\mathfrak{a}_i$ is a special direction of F; and if $\lambda_i \neq 0$, then $\mathfrak{a}_i$ is a nonspecial principal direction of F.

3. A hyperplane of symmetry of F = a principal diametral hyperplane (conjugate to a nonspecial principal direction). The intersection of a hyperplane of symmetry of F and the set F is equal to a set of vertices of F (if F has vertices).

4. Taking a proper centre (or a vertex) of F and principal directions of F (if necessary we can add the third direction perpendicular to two directions which we have) we construct an appropriate isometry. After putting to the equation of F we obtain a canonical equation.

Example 1. Classify the algebraic set $F : 2x_1^2 + x_2^2 - 4x_1x_2 + 2x_2 - 1 = 0$ in $\mathbb{R}^2$ from the affine point of view.

Solution. We determine improper points and singular points of F and answer the question. First, we complete the set F:

$$F^* : \varphi(x) = 2x_1^2 + x_2^2 - 4x_1x_2 + 2x_0x_2 - x_0^2 = 0 \;\text{ in }\; P^2.$$

Improper points:

We have to solve the following system (since improper points have the 0-coordinate $x_0 = 0$):

$$\begin{cases} x_0 = 0 \\ 2x_1^2 - 4x_1x_2 + x_2^2 = 0 \end{cases},$$

that is,

$$\begin{aligned}
&(4x_1^2 - 4x_1x_2 + x_2^2) - 2x_1^2 = 0 \\
&(2x_1 - x_2)^2 - 2x_1^2 = 0 \\
&\left(2x_1 - x_2 - \sqrt{2}x_1\right)\left(2x_1 - x_2 + \sqrt{2}x_1\right) = 0 \\
&x_2 = \left(2 - \sqrt{2}\right)x_1 \;\vee\; x_2 = \left(2 + \sqrt{2}\right)x_1.
\end{aligned}$$

Hence the set F has two improper points: $\{0, x_1, (2 - \sqrt{2})\,x_1\} = \{0, 1, 2 - \sqrt{2}\}$ and $\{0, x_1, (2 + \sqrt{2})\,x_1\} = \{0, 1, 2 + \sqrt{2}\}$.

Singular points:

$$\mathfrak{M}(\varphi) = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 2 & -2 \\ 1 & -2 & 1 \end{bmatrix} \quad \text{and} \quad \Delta(\varphi) = \begin{vmatrix} -1 & 0 & 1 \\ 0 & 2 & -2 \\ 1 & -2 & 1 \end{vmatrix} = -2 - 2 + 4 = 0.$$

Now, we know that F has at least one singular point $\Leftrightarrow \Delta(\varphi) = 0$. Hence F has singular points. We know that a point a is singular $\Leftrightarrow \varphi_i(a) = 0$ for $i = 0, 1, 2$. We have:

$$\begin{aligned}
\varphi_0(x) &= -x_0 + x_2 \\
\varphi_1(x) &= 2x_1 - 2x_2 \\
\varphi_2(x) &= x_0 - 2x_1 + x_2
\end{aligned}$$

(coefficients of the above are elements of rows of $\mathfrak{M}(\varphi)$)
and

$$\begin{cases} -x_0 + x_2 = 0 \\ 2x_1 - 2x_2 = 0 \\ x_0 - 2x_1 + x_2 = 0 \end{cases},$$

whence

$$\begin{cases} x_0 = x_2 \\ x_1 = x_2 \end{cases}.$$

The set F has one singular point: $\{x_2, x_2, x_2\} = \{1, 1, 1\}$.

Thus F is a pair of intersecting lines. $\square$

Example 2. Classify the algebraic set $F : 4x_1^2 - x_2^2 - 2x_3^2 - 16x_1 + 15 = 0$ in $\mathbb{R}^3$ from the affine point of view.

Solution. We complete the set F:

$$F^* : \varphi(x) = 4x_1^2 - x_2^2 - 2x_3^2 - 16x_0x_1 + 15x_0^2 = 0 \ \text{ in } \ P^3.$$

Improper points:

$$\begin{cases} x_0 = 0 \\ 4x_1^2 - x_2^2 - 2x_3^2 = 0 \end{cases}.$$

That is the equation of some conic in the improper plane.

Singular points:

$$\mathfrak{M}(\varphi) = \begin{bmatrix} 15 & -8 & 0 & 0 \\ -8 & 4 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \quad \text{and} \quad \Delta(\varphi) = \begin{vmatrix} 15 & -8 & 0 & 0 \\ -8 & 4 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{vmatrix} = -8 \neq 0.$$

Hence F does not have singular points. Thus F is a hyperboloid (we don't know which one, to find out we can check if it is a ruled set or we can find its canonical equation, see Example 4). $\square$

Example 3. Classify the algebraic set $F : x_1^2 + x_2^2 + x_3^2 + 2x_1x_3 - 2x_1 + 4x_2 + 4 = 0$ in $\mathbb{R}^3$ from the affine point of view.

Solution. We complete the set F:

$$F^* : \varphi(x) = x_1^2 + x_2^2 + x_3^2 + 2x_1x_3 - 2x_0x_1 + 4x_0x_2 + 4x_0^2 = 0 \ \text{ in } \ P^3.$$

Improper points:

$$\begin{cases} x_0 = 0 \\ x_1^2 + x_2^2 + x_3^2 + 2x_1x_3 = 0 \end{cases},$$

that is,

$$x_2^2 + (x_1 + x_3)^2 = 0$$
$$x_2 = 0 \ \wedge \ x_1 + x_3 = 0$$
$$x_2 = 0 \ \wedge \ x_3 = -x_1.$$

Hence the set F has one improper point: $\{0, x_1, 0, -x_1\} = \{0, 1, 0, -1\}$.

Singular points:

$$\mathfrak{M}(\varphi) = \begin{bmatrix} 4 & -1 & 2 & 0 \\ -1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \Delta(\varphi) = \begin{vmatrix} 4 & -1 & 2 & 0 \\ -1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{vmatrix} = -1 \neq 0.$$

Hence F does not have singular points. Thus F is an elliptic paraboloid. □

Example 4. Find the centre and principal directions of the algebraic set $F : 4x_1^2 - x_2^2 - 2x_3^2 - 16x_1 + 15 = 0$ in $\mathbb{R}^3$. Determine a canonical equation of F.

Solution. We see that F is the hyperboloid from Example 2. We have:

$$F^* : \varphi(x) = 4x_1^2 - x_2^2 - 2x_3^2 - 16x_0x_1 + 15x_0^2 = 0 \ \text{ in } \ P^3,$$

$$\mathfrak{M}(\varphi) = \begin{bmatrix} 15 & -8 & 0 & 0 \\ -8 & 4 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \quad \text{and} \quad \mathfrak{m}(\varphi) = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

The centre:

We know that F has precisely one proper centre $\Leftrightarrow \delta(\varphi) \neq 0$; and that F has at least one special direction (that is, an improper centre) $\Leftrightarrow \delta(\varphi) = 0$. We have

$$\delta(\varphi) = \det(\mathfrak{m}(\varphi)) = 8 \neq 0.$$

Hence the set F has precisely one proper centre. We know that a point a is a centre $\Leftrightarrow \varphi_i(a) = 0$ for $i = 1, 2, 3$. We have:

$$\begin{aligned}\varphi_1(x) &= -8x_0 + 4x_1\\ \varphi_2(x) &= -x_2\\ \varphi_3(x) &= -2x_3\end{aligned}$$

and

$$\begin{cases} -8x_0 + 4x_1 = 0 \\ -x_2 = 0 \\ -2x_3 = 0 \end{cases},$$

whence

$$\begin{cases} x_1 = 2x_0 \\ x_2 = 0 \\ x_3 = 0 \end{cases}.$$

Thus the point $\{x_0, 2x_0, 0, 0\} = \{1, 2, 0, 0\}$ is the proper centre of F with Cartesian coordinates: $(2, 0, 0)$.

Principal directions:
We know that eigenvectors of $\mathfrak{m}(\varphi)$ are principal directions of F. We have eigenvalues of $\mathfrak{m}(\varphi)$: $\lambda_1 = 4$, $\lambda_2 = -1$ and $\lambda_3 = -2$ and, respectively, eigenvectors of $\mathfrak{m}(\varphi)$: $\{0, x_1, 0, 0\} = \{0, 1, 0, 0\}$, $\{0, 0, x_2, 0\} = \{0, 0, 1, 0\}$ and $\{0, 0, 0, x_3\} = \{0, 0, 0, 1\}$. Hence these are nonspecial principal directions.

A canonical equation of F:
We have the centre $a = (2, 0, 0)$ and principal directions $\mathfrak{a}_1 = [1, 0, 0]$, $\mathfrak{a}_2 = [0, 1, 0]$ and $\mathfrak{a}_3 = [0, 0, 1]$. In order to find a canonical equation of F we have to write an isometry. We need a centre or a vertex of F and three versors. We have a centre and three versors, so the isometry has the form:

$$\begin{aligned}(x_1, x_2, x_3) &= a + \mathfrak{a}_1\overline{x}_1 + \mathfrak{a}_2\overline{x}_2 + \mathfrak{a}_3\overline{x}_3\\ &= (2, 0, 0) + [1, 0, 0]\overline{x}_1 + [0, 1, 0]\overline{x}_2 + [0, 0, 1]\overline{x}_3,\end{aligned}$$

that is,

$$\begin{cases} x_1 = 2 + \overline{x}_1 \\ x_2 = \overline{x}_2 \\ x_3 = \overline{x}_3 \end{cases}.$$

Setting the above to the equation of F we obtain the following canonical equation of F:

$$\frac{\overline{x}_2^2}{1} + \frac{\overline{x}_3^2}{\frac{1}{2}} - \frac{\overline{x}_1^2}{\frac{1}{4}} = -1.$$

Thus F is a hyperboloid of two sheets. $\square$

Example 5. Find the centre and principal directions of the algebraic set $F : x_3^2 - 3x_1 - 4x_2 - 5 = 0$ in $\mathbb{R}^3$. Determine a canonical equation of F.

Solution. We complete the set F:

$$F^* : \varphi(x) = x_3^2 - 3x_0x_1 - 4x_0x_2 - 5x_0^2 = 0 \ \text{ in } \ P^3.$$

So

$$\varphi(x) = 2x_3^2 - 6x_0x_1 - 8x_0x_2 - 10x_0^2 = 0,$$

$$\mathfrak{M}(\varphi) = \begin{bmatrix} -10 & -3 & -4 & 0 \\ -3 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad \mathfrak{m}(\varphi) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

The centre:

$$\delta(\varphi) = \det(\mathfrak{m}(\varphi)) = 0.$$

Hence the set F has special directions, that is, improper centres:

$$\begin{aligned} \varphi_1(x) &= -3x_0 \\ \varphi_2(x) &= -4x_0 \\ \varphi_3(x) &= 2x_3 \end{aligned}$$

and

$$\begin{cases} -3x_0 = 0 \\ -4x_0 = 0 \\ 2x_3 = 0 \end{cases},$$

whence

$$\begin{cases} x_0 = 0 \\ x_3 = 0 \end{cases}.$$

That is the improper line which contains all improper centres of F. Moreover the set F does not have a proper centre, so it can have a vertex.

Principal directions:

Eigenvalues of $\mathfrak{m}(\varphi)$: $\lambda_1 = 0$ and $\lambda_2 = 2$. For $\lambda_1 = 0$ we have already determined special (principal) directions (improper centres) of F. For $\lambda_2 = 2$ we have the nonspecial principal direction: $\{0, 0, 0, x_3\} = \{0, 0, 0, 1\}$.

The vertex:

The vertex is the intersection of F and a principal diametral hyperplane, that is, a diametral hyperplane conjugate to a nonspecial principal direction $a = \{0, 0, 0, 1\}$. We have

$$\begin{aligned}\varphi_0(x) &= -10x_0 - 3x_1 - 4x_2\\ \varphi_1(x) &= -3x_0\\ \varphi_2(x) &= -4x_0\\ \varphi_3(x) &= 2x_3\end{aligned}$$

and

$$\begin{aligned}\varphi_0(a) &= 0\\ \varphi_1(a) &= 0\\ \varphi_2(a) &= 0\\ \varphi_3(a) &= 2\end{aligned}.$$

Since the polar of the point a with respect to F has an equation

$$B(a) : \sum_{i=0}^{n} \varphi_i(a) \cdot x_i = 0,$$

we get

$$B(a) : 2x_3 = 0,$$

that is,

$$B(a) : x_3 = 0.$$

That is the principal diametral plane of F, that is, the plane of symmetry of F. The intersection of F and $B(a)$:

$$\begin{cases} x_3^2 - 3x_1 - 4x_2 - 5 = 0\\ x_3 = 0\end{cases},$$

that is,

$$\begin{cases} -3x_1 - 4x_2 - 5 = 0\\ x_3 = 0\end{cases}.$$

That is the line which contains all vertices of F. We choose one, for example, $b = (1, -2, 0)$.

A canonical equation of F:

In order to find a canonical equation of F yet we need three perpendicular versors associated with F. We have the first:

$$\mathfrak{a}_1 = [0, 0, 1].$$

As the second we take a singular direction of F. We have to solve the system:

$$\begin{cases} \varphi_0(x) = -10x_0 - 3x_1 - 4x_2 = 0 \\ \varphi_1(x) = -3x_0 = 0 \\ \varphi_2(x) = -4x_0 = 0 \\ \varphi_3(x) = 2x_3 = 0 \end{cases},$$

that is,

$$\begin{cases} x_0 = 0 \\ x_3 = 0 \\ x_1 = -\frac{4}{3}x_2 \end{cases}.$$

Hence $\left\{0, -\frac{4}{3}x_2, x_2, 0\right\} = \{0, -4, 3, 0\}$ is the singular direction of F. So we have the vector $[-4, 3, 0]$ and the second versor:

$$\mathfrak{a}_2 = \frac{[-4, 3, 0]}{|[-4, 3, 0]|} = \left[-\frac{4}{5}, \frac{3}{5}, 0\right].$$

As the third versor we take:

$$\mathfrak{a}_1 \times \mathfrak{a}_2 = \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ -\frac{4}{5} & \frac{3}{5} & 0 \end{vmatrix} = \left[-\frac{3}{5}, -\frac{4}{5}, 0\right] \parallel \left[\frac{3}{5}, \frac{4}{5}, 0\right] = \mathfrak{a}_3.$$

Hence the isometry has the form:

$$\begin{aligned}(x_1, x_2, x_3) &= b + \mathfrak{a}_1\overline{x}_1 + \mathfrak{a}_2\overline{x}_2 + \mathfrak{a}_3\overline{x}_3 \\ &= (1, -2, 0) + [0, 0, 1]\overline{x}_1 + \left[-\frac{4}{5}, \frac{3}{5}, 0\right]\overline{x}_2 + \left[\frac{3}{5}, \frac{4}{5}, 0\right]\overline{x}_3,\end{aligned}$$

that is,

$$\begin{cases} x_1 = 1 - \frac{4}{5}\overline{x}_2 + \frac{3}{5}\overline{x}_3 \\ x_2 = -2 + \frac{3}{5}\overline{x}_2 + \frac{4}{5}\overline{x}_3 \\ x_3 = \overline{x}_1 \end{cases}.$$

Setting the above to the equation of F we obtain the following canonical equation of F:

$$\overline{x}_1^2 - 5\overline{x}_3 = 0.$$

Thus F is a parabolic cylinder. $\square$

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